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Computing the Brauer–Long group of a Hopf algebra II: The Skolem–Noether theory

S. Caenepeel

Faculty of Applied Sciences, Free University of Brussels, VUB, Pleinlaan 2, B-1050 Brussels, Belgium

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Abstract

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Let H be a commutative, cocommutative and faithfully projective Hopf algebra over a commutative ring R. Then Long's Brauer group of H-dimodule algebras fits into an exact sequence $1 \to \mathrm{BD}^{s}(R, H) \to \mathrm{BD}(R, H) \xrightarrow{\beta} O(R, H)_{\min}$, where $O(R, H)_{\min}$ is a well-defined subgroup of the group of Hopf algebra automorphisms of $H \otimes H^{*}$. It is not known whether β is surjective, but some partial results are given. The theory is applied to Orzech's and Deegan's subgroups of the Brauer–Long group.

Introduction

Let R be a commutative ring, and H a commutative, cocommutative and faithfully projective Hopf algebra. Long [27, 28] introduced a Brauer group $\mathrm{BD}(R, H)$, which turned out to be a very natural generalization of the Brauer–Wall group (cf. [3, 11, 31, 36]). Long's Brauer group, and related Brauer groups, have been studied by several authors in the case where the Hopf algebra H is a groupring RG (G a finite abelian group); Let us mention [4, 6, 7, 16, 18, 19, 24, 29] (the list is far from complete). Recently, a complete description was given in the case where G is cyclic group of primary order (cf. [8, 20]). In [14], the author and Beattie proposed a general approach; they could show that, under the condition that $|G|$ is invertible in R and R contains 'enough' roots of unity, the

Correspondence to: Professor S. Caenepeel, Faculty of Applied Sciences, Free University of Brussels, VUB, Pleinlaan 2, B-1050 Brussels, Belgium.

Brauer–Long group is described by an exact sequence

$$(*)\qquad 1 \to \mathrm{BD}^s(R, G) \to \mathrm{BD}(R, G) \xrightarrow{\beta} O(R, G)_{\min} \to 1\,.$$

Here $\mathrm{BD}^s(R, G)$ is a product of the Brauer group of R and two copies of the group of G-Galois extensions of R, and $O(R, G)_{\min}$ is a well-defined orthogonal subgroup of $\mathrm{Aut}(G \times G^*)$. The aim of this paper, and the preceding paper [10] is to generalize $(*)$ to the Brauer–Long group of a Hopf algebra. In [10], the first half of the project was carried out: using cohomological methods, it was shown that $\mathrm{BD}^s(R, H)$ is a semi-direct product of the Brauer group and the groups of H-Galois objects and H^*-Galois objects in the sense of Sweedler. In this paper, we will construct a generalization of the map β. It is not very difficult to show that we have a map

$$\beta : \mathrm{BD}(R, H) \to \mathrm{Aut}(G(H \otimes H^*))\,,$$

but this happens to be insufficient to fit $\mathrm{BD}(R, H)$ into a short exact sequence.

Applying the so-called 'tensoring-up trick' introduced in [12], we may define a map

$$\beta : \mathrm{BD}(R, H) \to \mathrm{Aut}_{\mathrm{Hopf}}(H \otimes H^*)\,.$$

This construction is discussed in Section 2; in Section 3, it is shown that $\mathrm{Ker}\,\beta = \mathrm{BD}^s(R, G)$. It may be shown easily that $\mathrm{Im}\,\beta$ is a subgroup of $O(R, H)_{\min}$, a well-defined 'orthogonal' subgroup of $\mathrm{Aut}_{\mathrm{Hopf}}(H \otimes H^*)$, but it is not known whether β is surjective onto $O(R, H)_{\min}$. In Section 4 we show that $\mathrm{Aut}_{\mathrm{Hopf}}(H)$ may be embedded as a subgroup of $\mathrm{BD}(R, H)$. If $H = RG$, then this was studied by Deegan in [19].

In Section 5 we show that H itself represents an element of $\mathrm{BD}(R, H)$, if there exists a selfdual Hopf algebra isomorphism $f : H \to H^*$. Section 6 applies our theory to a subgroup of $\mathrm{BD}(R, H)$ defined by Orzech in [30]. Finally, in Section 7, we compute the Brauer–Long group of a Hopf algebra of rank two, and prove some results announced in [10].

1. Notations and preliminary results

For all unexplained terminology, we refer to [10]. In this section, we will recall some of the results of [10] that will be used in the sequel. Throughout this paper, R will be a commutative ring, and H a commutative, cocommutative and finite (that is, a finitely generated, faithful and projective) Hopf algebra over R. J will be a shorter notation for the Hopf algebra $J = H \otimes H^*$. The H-dimodule isomorphism classes of invertible H-dimodules form a group under the tensor product, denoted by $\mathrm{PD}(R, H)$. We have the following property:

1.1. Proposition [10, Proposition 1.6].

$$\mathrm{PD}(R, H) \cong \mathrm{Pic}(R) \times G(J) \cong \mathrm{Pic}(R) \times G(H) \times G(H^*) . \qquad \square$$

Recall that the element of $\mathrm{PD}(R, H)$ coresponding to $([I], h, h^*) \in \mathrm{Pic}(R) \times G(H) \times G(H^*)$ is represented by $I(h, h^*)$, which is the H-dimodule equal to I as an R-module, and with dimodule structure given by

$$\chi(x) = x \otimes h \qquad \text{for all } x \in I ,$$
$$k \rightarrow x = h^*(k)x \quad \text{for all } x \in I, k \in H .$$

1.2. *H*-Azumaya algebras and the Brauer–Long group. An H-dimodule algebra A is called *H-Azumaya* if A is faithfully projective as an R-module and if the homomorphisms

$$F : A\#\bar{A} \rightarrow \mathrm{End}_R(A) \quad \text{and} \quad G : \bar{A}\#A \rightarrow \mathrm{End}_R(A)^{\mathrm{opp}}$$

defined by

$$F(a\#b)(c) = \sum_{(b)} a(b_{(1)} \rightarrow c)b_{(0)} , \qquad G(a\#b)(c) = \sum_{(c)} (c_{(1)} \rightarrow a)c_{(0)}b$$

are isomorphisms. Here $\#$ is the smash product, and $\bar{A}$ is the H-opposite algebra of A. The smash product $A\#B$ of two H-dimodule algebras is $A \otimes B$ as an H-dimodule, with multiplication rule given by

$$(a\#b)(c\#d) = \sum_{(b)} a(b_{(1)} \rightarrow c)\#b_{(0)}d \quad (a,c \in A; b,d \in B) .$$

Similarly, $\bar{A}$ is A as an H-dimodule, with multiplication

$$\bar{a}.\bar{b} = \overline{\sum_{(a)} (a_{(1)} \rightarrow b)a_{(0)}} \quad (a,b \in A) .$$

Two H-Azumaya algebras A and B are called H-equivalent if there exist faithfully projective H-dimodules P,Q such that

$$A\#\mathrm{End}_R(P) \cong B\#\mathrm{End}_R(Q)$$

as H-dimodule algebras. Long [28] has shown that the H-equivalence classes of isomorphism classes of H-Azumaya algebras form a group under the operation induced by the smash product. Nowadays, this group is called the Brauer–Long group, and it is denoted by $\mathrm{BD}(R, H)$. $\mathrm{BD}(-, -\otimes \mathrm{H})$ defines a functor from commutative R-algebras to abelian groups; the kernel of the map $\mathrm{BD}(R, H) \rightarrow \mathrm{BD}(S, S \otimes H)$ is denoted by $\mathrm{BD}(S/R, H)$. We write

$$\mathrm{BD}^{\mathrm{s}}(R, H) = \bigcup \mathrm{BD}(S/R, H)\,,$$

the union being taken over all faithfully flat R-algebras. A full cohomological description of $\mathrm{BD}^{\mathrm{s}}(R, H)$ has been given in [10, Theorem 3.9]; it turned out that $\mathrm{BD}^{\mathrm{s}}(R, H)$ is a semi-direct product of $\mathrm{Br}(R)$ and $\mathrm{Gal}^{\mathrm{s}}(R, H) \times \mathrm{Gal}^{\mathrm{s}}(R, H^*)$, the groups of commutative H and H^*-Galois objects.

Suppose that A is an H-dimodule algebra. We introduce the following notations:

$$H\text{-}\mathrm{Aut}_R(A)$$
$$= \{f : A \to A \mid f \text{ is an } H\text{-dimodule algebra automorphism}\}\,,$$

$$H\text{-}\mathrm{INN}_R(A)$$
$$= \{f \in H\text{-}\mathrm{Aut}_R(A) \mid f \text{ is induced by } x \in A^H \text{ satisfying } \chi(x) = x \otimes 1\}.$$

Proposition 1.3 generalizes the Skolem–Noether theorem; the proof is a straightforward generalization of [25, IV.1.2].

1.3. Proposition [12]. *If A is an H-Azumaya algebra, then we have an exact sequence*

$$1 \to H\text{-}\mathrm{INN}_R(A) \to H\text{-}\mathrm{Aut}_R(A) \xrightarrow{\Phi} \mathrm{PD}(R, H)\,.$$

Here Φ is defined by $\Phi(f) = [I_f]$, where $I_f = \{x \in A \mid \sum_{(a)} (a_{(1)} \to x)a_{(0)} = f(a)x$, for all $a \in A\}$. □

1.4. The Hopf algebra $\mathcal{H} = \mathrm{Hom}_R(K, H)$. Let K,H be Hopf algebras that are commutative, cocommutative and finite. In [12], it was shown that $\mathcal{H} = \mathrm{Hom}_R(K, H) \cong \underline{K}^* \otimes H$ may be given the structure of a Hopf algebra over the base ring $\underline{K}^*$ as follows (by $\underline{K}^*$, we mean K^* as a commutative R-algebra, that is, we forget the coalgebra structure on K^*):

$$\eta_{\mathcal{H}} : \underline{K}^* \to \mathcal{H} \quad \text{and} \quad \varepsilon_{\mathcal{H}} : \mathcal{H} \to \underline{K}^*$$

$$\text{given by} \quad \eta_{\mathcal{H}}(k^*) = \eta_H \circ k^* \quad \text{and} \quad \varepsilon_{\mathcal{H}}(\mu) = \varepsilon_H \circ \mu\,;$$

$$\Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes_{\underline{K}^*} \mathcal{H} \cong \mathrm{Hom}_R(K, H \otimes H)$$

$$\text{given by} \quad \Delta_{\mathcal{H}}(\mu)(k) = \Delta_H(\mu(k))\,;$$

$$m_{\mathcal{H}} : \mathcal{H} \otimes_{\underline{K}^*} \mathcal{H} \to \mathcal{H}$$

$$\text{given by} \quad m_{\mathcal{H}}(\mu \otimes \nu) = \mu * \nu = m_H \circ (\mu \otimes \nu) \circ \Delta_K\,;$$

and finally

$$S_{\mathscr{H}} : \mathscr{H} \to \mathscr{H} \quad \text{given by} \quad S_{\mathscr{H}}(\mu) = S_H \circ \mu \ .$$

If A is an H-Azumaya algebra, then $\mathscr{A} = \mathrm{Hom}_R(K, A) \cong \underline{K}^* \otimes A$ is an $\mathscr{H}$-Azumaya algebra. The structure maps are given by

$$\chi_{\mathscr{A}} : \mathscr{A} \to \mathscr{A} \otimes_{\underline{K}^*} \mathscr{H} \cong \mathrm{Hom}_R(K, A \otimes H) \ , \qquad \chi_{\mathscr{A}}(u) = \chi_A \circ u$$

and

$$(\mu \to u)(k) = \sum_{(k)} \mu(k_{(1)}) \to u(k_{(2)}).$$

From [12], we recall the following result:

1.5. Proposition. *With notations as above, we have*

$$G(\mathscr{H}) = \{\mu \in \mathscr{H} \colon \mu \text{ is a coalgebra homomorphism}\} \ . \qquad \square$$

1.6. Integrals. Recall from [33, Chapter V] that an element $x^* \in H^*$ is called an integral if for all $h^* \in H^*$, we have that $x^* * h^* = \langle h^*, 1\rangle x$. If we denote $\int = \{x^* \in H^* \colon x^* \text{ is an integral}\}$, then we have the following result:

1.7. Proposition. *Let H be a finite, commutative and cocommutative Hopf algebra. Then the map $V : \int \otimes H \to H^*$, defined by $\langle V(x^* \otimes h), l\rangle = \langle x^*, S(h)l\rangle$ is an isomorphism of R-modules. Consequently $\int$ is a projective R-module of rank one.*

Proof. In [32, V.1.3], this is shown for a Hopf algebra over a field. There is no problem to generalize Sweedler's proof to the case of a commutative ring. $\square$

The next lemma will be used in the proof of Lemma 6.2.

1.8. Lemma. *Fix $x^* \in \int$, and let $v : H \to H^*$ be defined by $v(h) = V(x^* \otimes h)$, for $h \in H$. Then v satisfies the identity*

$$\sum_{(k)} \langle v(h), k_{(1)}\rangle k_{(2)} = \sum_{(h)} \langle v(h_{(1)}), k\rangle h_{(2)} \ ,$$

for all $h, k \in H$.

Proof. It suffices to show that for all $h^* \in H^*$, we have

$$\sum_{(k)} \langle v(h), k_{(1)}\rangle \langle h^*, k_{(2)}\rangle = \sum_{(h)} \langle v(h_{(1)}), k\rangle \langle h^*, h_{(2)}\rangle \ .$$

Indeed,

$$\begin{aligned}
&\sum_{(k)} \langle v(h), k_{(1)}\rangle \langle h^*, k_{(2)}\rangle \\
&\quad = \sum_{(k)} \langle x^*, S(h)k_{(1)}\rangle \langle h^*, k_{(2)}\rangle \\
&\quad = \sum_{(k),(h^*),(h)} \langle x^*, S(h_{(1)})k_{(1)}\rangle \langle h^*_{(1)}, h_{(1)}\rangle \langle h^*_{(2)}, S(h_{(2)})k_{(1)}\rangle \\
&\quad = \sum_{(h^*),(h)} \langle h^*_{(1)}, h_{(1)}\rangle \langle h^*_{(2)} * x^*, S(h_{(2)})k\rangle \\
&\quad = \sum_{(h^*),(h)} \langle h^*_{(1)}, h_{(1)}\rangle \langle h^*_{(2)}, 1\rangle \langle x^*, S(h_{(2)})k\rangle \\
&\quad = \sum_{(h)} \langle h^*, h_{(1)}\rangle \langle x^*, S(h_{(2)})k\rangle .
\end{aligned}$$

Remark that, in the case where $H = RG$ is a groupring, the map v is defined by $v(u_\sigma) = v_\sigma$, where $\{u_\sigma : \sigma \in G\}$ is the canonical basis of the groupring, and $\{v_\sigma : \sigma \in G\}$ is the canonical basis for $(RG)^*$, which is given by $\langle v_\sigma, u_\tau\rangle = \delta_{\sigma,\tau}$, for all $\sigma, \tau \in G$. □

2. Definition of the map β

2.1. The groups $O(G(J))_{\min}$ and $O(G(J))_{\max}$. Recall that $J = H \otimes H^*$; therefore we have that $J^* = H^* \otimes H$. In the sequel, we will frequently use the bracket notation, that is, for $h \in H$, $h^* \in H^*$, we write $h^*(h) = \langle h, h^*\rangle$. Define a Hopf algebra map $\varphi : J \to J^*$ by

$$\varphi(h \otimes h^*) = h^* \otimes \eta\varepsilon(h) .$$

It is easily seen that the dual map $\varphi^* : J \to J^*$ is then given by (we identify J and J^{**} by the canonical isomorphism)

$$\varphi^*(h \otimes h^*) = \varepsilon^*\eta^*(h^*) \otimes h ,$$

and that the convolution $t = \varphi * \varphi^*$ is the switch map

$$t(h \otimes h^*) = h^* \otimes h .$$

The Hopf algebra maps φ, φ^* and t restrict to group homomorphisms $G(J) \to G(J^*)$. For example, for $h \otimes h^* \in G(J)$, we have that $\varphi(h \otimes h^*) = h^* \otimes 1$.

A map $q : J \to R$ is defined by $q(h \otimes h^*) = \langle h^*, h \rangle$, for all $h \in H$, $h^* \in H^*$. A map $b : J \otimes J \to R$ is defined by $b(j \otimes k) = \langle j, t(k) \rangle$, for all $j,k \in J$, that is, $b(h \otimes h^*, k \otimes k^*) = \langle h^*, k \rangle \langle k^*, h \rangle$. Observe that q and b restrict to maps

$$q : G(J) \to \mathbb{G}_{\mathrm{m}}(R) \quad \text{and} \quad b : G(J) \times G(J) \to \mathbb{G}_{\mathrm{m}}(R) .$$

Define the following 'orthogonal' subgroups of $\mathrm{Aut}(G(J))$:

$$O(G(J))_{\min} = \{ f \in \mathrm{Aut}(G(J)) \colon q \circ f = q \} ,$$

$$\begin{aligned} O(G(J))_{\max} &= \{ f \in \mathrm{Aut}(G(J)) \colon b \circ (f, f) = b \} \\ &= \{ f \in \mathrm{Aut}(G(J)) \colon f^* \circ t \circ f = t \} . \end{aligned}$$

Consider an H-Azumaya algebra A. We define a map

$$\rho = \rho_1 \otimes \rho_2 : J = H \otimes H^* \to \mathrm{End}_R(A)$$

as follows

$$\rho_1(h)(a) = h \to a , \qquad \rho_2(h^*)(a) = \sum_{(a)} h^*(a_{(1)}) a_{(0)} ,$$

for all $h \in H$, $h^* \in H^*$, $a \in A$. ρ restricts to a map $\rho : G(J) \to H\text{-}\mathrm{Aut}_R(A)$.

Applying Propositions 1.1 and 1.3 we define the map α_A to be the following composition:

$$\alpha_A : G(J) \xrightarrow{\rho} H\text{-}\mathrm{Aut}_R(A) \xrightarrow{\Phi} \mathrm{PD}(R, H) \to G(J) .$$

The map $\beta_A : G(J) \to G(J)$ is defined by

$$\beta_A(j) = j(\alpha_A(j))^{-1} .$$

2.2. Theorem. *With notations as above, we have*:

(1) α_A, β_A *do not depend upon the choice of* A *in* $[A]$.

(2) β_A *is an automorphism of* $G(J)$; *moreover*, $\beta_A \in O(G(J))_{\min}$.

(3) *The map* $\beta : \mathrm{BD}(R, H) \to O(G(J))_{\min}$, *defined by* $\beta([A]) = \beta_A$ *is a group homomorphism*; *we therefore have that* $\beta_{A\#B} = \beta_A \circ \beta_B$, $\alpha_{A\#B} = (\alpha_A \circ \beta_B) * \alpha_A$.

(4) $\mathrm{BD}^{\mathrm{s}}(R, H) \subset \mathrm{Ker}\, \beta$.

(5) $\mathrm{Im}\, \beta \subset O(G(J))_{\min} \subset O(G(J))_{\max}$.

Proof. The proof of Theorem 2.2 is an immediate generalization of the proofs of Theorem 3.2, Proposition 3.3 and part (3.4.1) of the proof of Theorem 3.4 in [14]. We leave verification and details to the reader. □

From Theorem 2.2, it follows that we have a homomorphism $\bar{\beta} : \mathrm{BD}(R, H)/\mathrm{BD}^s(R, H) \to O(G(J))_{\min}$. If H is a groupring, that is, $H = RG$, and if R contains a primitive $\exp(G)$-th root of 1, and if $|G|$ is invertible in R, then it may be shown that $\bar{\beta}$ is an isomorphism (cf. [14, Theorem 3.4]). In general, this property does not hold, unfortunately. The aim of this paper is to define a map $\beta : \mathrm{BD}(R, H) \to \mathrm{Aut}_{\mathrm{Hopf}}(J)$; to this end, we will extend the group homomorphism $\beta_A : G(J) \to G(J)$ defined above to a Hopf algebra automorphism $\beta_A : J \to J$. We will apply the 'tensoring-up trick' introduced in [12] to study inner actions of Hopf algebras on Azumaya algebras and H-Azumaya algebras.

In the construction 1.4, replace the Hopf algebra K by $J^{(n)} = J \otimes J \otimes \cdots \otimes J$, and write

$$\mathcal{H}_n = \mathrm{Hom}_R(J^{(n)}, H) \cong J^{*(n)} \otimes H .$$

In the sequel, we will use $\mathcal{H}_1 = \mathcal{H}$, and $\mathcal{H}_2$. For $n = 1$, we will omit the index n. $\mathcal{H}_2$ will only be used in the (technical) Lemmas 2.6 and 2.7, so, for simplicity, we recommend the reader to forget about the index n at a first reading of this paper. Observe that

$$\mathcal{H}_n^* \cong \mathrm{Hom}_R(J^{(n)}, H^*) \quad \text{and} \quad \mathcal{J}_n = \mathcal{H}_n \otimes_{J^{*(n)}} \mathcal{H}_n^* \cong \mathrm{Hom}_R(J^{(n)}, J) .$$

Let A be an H-Azumaya algebra; then

$$\mathcal{A}_n = \mathrm{Hom}_R(J^{(n)}, A) \cong \underline{J}^{*(n)} \otimes A$$

is an $\mathcal{H}_n$-Azumaya algebra. The construction preceding Theorem 2.2 gives maps

$$\alpha_{\mathcal{A}_n}, \beta_{\mathcal{A}_n} : G(\mathcal{J}_n) \to G(\mathcal{J}_n) .$$

Let us describe $\alpha_{\mathcal{A}_n}$ and $\beta_{\mathcal{A}_n}$ explicitly. We have

$$\alpha_{\mathcal{A}_n} : G(\mathcal{J}_n) \xrightarrow{F} \mathcal{H}_n\text{-}\mathrm{Aut}_{J^{*(n)}}(\mathcal{A}_n) \xrightarrow{\Phi} \mathrm{PD}(\underline{J}^{*(n)}, \mathcal{H}_n) \to G(\mathcal{J}_n) , \tag{1}$$

where F is given by

$$F(\gamma)(f)(j) = \sum_{(j)} \rho(\gamma(j_{(1)}))(f(j_{(2)})) \tag{2}$$

for all $j \in J^{(n)}$, $\gamma \in G(\mathcal{J}_n)$, $f \in \mathcal{A}_n$. ρ was defined in 2.1. $\beta_{\mathcal{A}_n}$ is defined by

$$\beta_{\mathcal{A}_n}(\gamma) = \gamma * (S \circ \alpha_{\mathcal{A}_n}(\gamma)) .$$

Recall from Proposition 1.3 that, for $\mathcal{F} \in \mathcal{H}_n\text{-}\mathrm{Aut}_{J^{*(n)}}(\mathcal{A}_n)$, $\Phi(\mathcal{F})$ is represented by an invertible $\mathcal{H}_n$-dimodule denoted by $I_{\mathcal{F}}$, which is a submodule of $\mathcal{A}_n$. For $\mathcal{F} = F(\gamma)$, a description of $I_{F(\gamma)}$ is given in the following lemma:

2.3. Lemma. *With notations as above, let* $u \in \mathcal{A}_n$, *and suppose that* $\alpha_{\mathcal{A}_n}(\gamma) = \mu \otimes \mu^*$, *with* $\mu \in G(\mathcal{H}_n)$, $\mu^* \in G(\mathcal{H}_n^*)$. *Then* $u = I_{F(\gamma)}$ *if and only if for all* $j \in J^{(n)}$, $a \in A$, *we have*:

$$\sum_{(a),(j)} \langle a_{(1)}, \mu^*(j_{(1)})\rangle u(j_{(2)})a_{(0)} = \sum_{(j)} \rho(\gamma(j_{(1)}))(a)u(j_{(2)})\,. \tag{3}$$

Proof. First suppose that $u \in I_{F(\gamma)}$. By Proposition 1.3, we have for all $f \in \mathcal{A}_n$ that

$$\sum_{(f)} (f_{(1)} \rightarrow u) * f_{(0)} = F(\gamma)(f) * u$$

or

$$\sum_{(f)} \langle f_{(1)}, \mu^*\rangle * u * f_{(0)} = F(\gamma)(f) * u\,. \tag{4}$$

Let $i : A \rightarrow \mathcal{A}_n$ be the natural embedding. Then, for $a \in A$, $i(a)$ is given by $i(a)(j) = \varepsilon(j)a$, for all $j \in J^{(n)}$ (cf. [12]). In (4), replace f by $i(a)$; then we obtain that, for all $a \in A$, $j \in J^{(n)}$,

$$\begin{aligned}\sum_{(a),(j)} &\langle i(a_{(1)}), \mu^*\rangle(j_{(1)})u(j_{(2)})i(a_{(0)})(j_{(3)})\\ &= \sum_{(j)} F(\gamma)(i(a))(j_{(1)})u(j_{(2)})\\ &= \sum_{(j)} \rho(\gamma(j_{(1)}))(i(a)(j_{(2)}))u(j_{(3)})\\ &= \sum_{(j)} \rho(\gamma(j_{(1)}))(a)u(j_{(2)})\end{aligned}$$

or

$$\sum_{(a),(j)} \langle a_{(1)}, \mu^*(j_{(1)})\rangle u(j_{(2)})a_{(0)} = \sum_{(j)} \rho(\gamma(j_{(1)}))(a)u(j_{(2)})\,.$$

Conversely, suppose that (3) holds for all $a \in A$. Then we have to show that (4) holds for all $f \in \mathcal{A}_n$. Since $J^{(n)}$ is projective as an R-module, we have a dual basis $\{j_1, \ldots, j_q; j_1^*, \ldots, j_q^*\}$ for $J^{(n)}$. This means that for all $j \in J^{(n)}$, we have that

$$j = \sum_{i=1}^{q} j_i^*(j)j_i\,.$$

Therefore every $f \in \mathscr{A}_n$ may be written as

$$f(-) = \sum_{i=1}^{q} j_i^*(-)f(j_i) \quad \text{or} \quad f = \sum_{i=1}^{q} j_i^* * i(f(j_i)) .$$

Since (4) holds for f of the form $i(a)$, it follows by linearity that it holds for every f. $\square$

Our next lemma shows that $\alpha_{\mathscr{A}_n}$ and $\beta_{\mathscr{A}_n}$ are known completely once we know $\alpha_{\mathscr{A}}(I)$ or $\beta_{\mathscr{A}}(I)$. Here $I : J \to J$, the identity of J, is a grouplike element of $\mathscr{J} = \mathrm{Hom}_R(J, J)$. Recall that for $n = 1$, we omit the index n.

2.4. Lemma. *With notations as above, we have that, for all $\gamma \in G(\mathscr{J}_n)$,*

$$\beta_{\mathscr{A}_n}(\gamma) = \beta_{\mathscr{A}}(I) \circ \gamma , \qquad \alpha_{\mathscr{A}_n}(\gamma) = \alpha_{\mathscr{A}}(I) \circ \gamma .$$

Proof. From (2), it follows that, for all $j \in J^{(n)}$, $\gamma \in G(\mathscr{J}_n)$, $f \in \mathscr{A}$,

$$\begin{aligned} F(I)(f)(\gamma(j)) &= \sum_{(j)} \rho(\gamma(j_{(1)}))(f(\gamma(j_{(2)}))) \\ &= \sum_{\gamma(j)} \rho(\gamma(j)_{(1)})(f(\gamma(j)_{(2)})) \\ &= F(\gamma)(f \circ \gamma)(j) , \end{aligned}$$

where we used the fact that γ is grouplike, and therefore a coalgebra homomorphism. It follows that $F(I)(f) \circ \gamma = F(\gamma)(f \circ \gamma)$. Take $u \in I_{F(I)}$, and suppose that $\alpha_{\mathscr{A}}(I) = \mu \otimes \mu^*$. Then for all $a \in A$ and $j \in J$,

$$\sum_{(a),(j)} \langle a_{(1)}, \mu^*(j_{(1)}) \rangle u(j_{(2)}) a_{(0)} = \sum_{(j)} \rho(j_{(1)})(a) u(j_{(2)}) .$$

Replacing j by $\gamma \circ j$ (where $j \in J^{(n)}$), and taking into account that j is a coalgebra homomorphism, we obtain

$$\begin{aligned} &\sum_{(a),(j)} \langle a_{(1)}, (\mu^* \circ \gamma)(j_{(1)}) \rangle (u \circ \gamma)(j_{(2)}) a_{(0)} \\ &\quad = \sum_{(j)} \rho(\gamma(j_{(1)}))(a)(u \circ \gamma)(j_{(2)}) . \end{aligned}$$

From Lemma 2.3, it now follows that $u \circ \gamma \in I_{F(\gamma)}$, and therefore

$$\alpha_{\mathscr{A}_n}(\gamma) = (\mu \circ \gamma) \otimes (\mu^* \circ \gamma) = \alpha_{\mathscr{A}}(I) \circ \gamma .$$

Finally,

$$\beta_{\mathscr{A}_n}(\gamma) = \gamma * S_{\mathscr{J}_n}(\alpha_{\mathscr{A}_n}(\gamma)) = \gamma * S_J \circ \alpha_{\mathscr{A}}(I) \circ \gamma$$
$$= (I * S_J \circ \alpha_{\mathscr{A}}(I)) \circ \gamma = \beta_{\mathscr{A}}(I) \circ \gamma\ .$$

This finishes the proof of the lemma. □

$\alpha_{\mathscr{A}}(I)$ and $\beta_{\mathscr{A}}(I)$ are the promised extensions of α_A and β_A to J. Let us first show that their restrictions to $G(J)$ are indeed α_A and β_A.

2.5. Lemma. $\alpha_{\mathscr{A}}(I)|_{G(J)} = \alpha_A$, $\beta_{\mathscr{A}}(I)|_{G(J)} = \beta_A$.

Proof. Take $j \in G(J)$, and $u \in I_{F(I)}$. For all $a \in A$, we have, by Lemma 2.3,

$$\sum_{(a)} \langle a_{(1)}, \mu^*(j)\rangle u(j) a_{(0)} = \rho(j)(a) u(j)\ ,$$

where we denoted $\alpha_{\mathscr{A}}(I) = \mu \otimes \mu^*$, as before. Suppose that $\alpha_A(j) = h \otimes h^*$. Then on one hand, we have that for all $k \in G(H)$, $k \rightarrow u(j) = \langle k, h^*\rangle u(j)$, and on the other hand that $\nu \rightarrow u = \langle \nu, \mu^*\rangle * u$, for all $\nu \in G(\mathscr{H})$. Taking $\nu = i(k)$, with $k \in G(H)$, and applying both sides to j we obtain

$$(i(k) \rightarrow u)(j) = k \rightarrow u(j) = \langle k, \mu^*(j)\rangle u(j)\ ,$$

and it follows that $h^* = \mu^*(j)$. A similar computation shows that $h = \mu(j)$, and this finishes the proof of the lemma. □

From now on, we will write β_A also for the extended $\beta_A = \beta_{\mathscr{A}}(I) : J \rightarrow J$. Now reconsider the maps $\varphi, \varphi^*, t : J \rightarrow J^*$ defined in 2.1. They extend to maps $\Phi, \Phi^*, T : \mathscr{J}_n \rightarrow \mathscr{J}_n^*$. The relations between φ, φ^*, t and Φ, Φ^*, T are given by the formulas

$$\Phi(\mu) = \varphi \circ \mu\ , \quad \Phi^*(\mu) = \varphi^* \circ \mu\ , \quad T(\mu) = t \circ \mu\ , \qquad \text{for all } \mu \in \mathscr{J}_n\ .$$

$Q : \mathscr{J}_n \rightarrow \underline{J}^*$ is defined by $Q(\mu) = \langle \mu, \Phi(\mu)\rangle \in \underline{J}^*$, or $Q(\mu) = \sum_{(j)} \langle \mu(j_{(1)}), (\varphi \circ \mu)(j_{(2)})\rangle$. If we apply Theorem 2.2(2) to the $\mathscr{H}_n$-Azumaya algebra $\mathscr{A}_n$, then it follows that, for all $\mu \in G(\mathscr{J}_n)$:

$$Q(\mu) = Q(\beta_{\mathscr{A}_n}(\mu))\ , \tag{5}$$

$$\beta^*_{\mathscr{A}_n} \circ T \circ \beta_{\mathscr{A}_n} = T\ . \tag{6}$$

2.6. Lemma. *Let A be an H-Azumaya algebra. Then $\beta_A = \beta_{\mathscr{A}}(I) = f$ satisfies the identities $q = q \circ f$ and $t = f^* \circ t \circ f$.*

Proof. From (5) it follows that $Q(I) = Q(f)$. Thus for all $j \in J$,

$$Q(I)(j) = \sum_{(j)} \langle j_{(1)}, \varphi(j_{(2)}) \rangle = q(j)$$

equals

$$Q(f)(j) = \sum_{(j)} \langle f(j_{(1)}), \varphi(f(j_{(2)})) \rangle = q(f(j)),$$

and this shows that $q = q \circ f$.

In order to show that $t = f^* \circ t \circ f$, we will need the construction following Theorem 2.2 in the case where $n = 2$. From (6) it follows that, for all $\mu, \nu \in G(\mathcal{J}_2)$,

$$\begin{aligned} \langle \beta_{\mathcal{A}_2}(\mu), (T \circ \beta_{\mathcal{A}_2})(\nu) \rangle &= \langle \mu, (\beta^*_{\mathcal{A}_2} \circ T \circ \beta_{\mathcal{A}_2})(\nu) \rangle \\ &= \langle \mu, T(\nu) \rangle = \langle \mu, t \circ \nu \rangle . \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \beta_{\mathcal{A}_2}(\mu), (T \circ \beta_{\mathcal{A}_2})(\nu) \rangle &= \langle \beta_{\mathcal{A}_2}(\mu), T(\beta_{\mathcal{A}_2})(\nu) \rangle \\ &= \langle \mu, \beta^*_{\mathcal{A}_2}(T(\beta_{\mathcal{A}_2}(\nu))) \rangle . \end{aligned}$$

Take $\mu = I \otimes \varepsilon$, $\nu = \varepsilon \otimes I$; it is clear that $\beta_{\mathcal{A}_2}(\mu) = \beta_{\mathcal{A}}(I) \otimes \varepsilon$ and $\beta_{\mathcal{A}_2}(\mu) = \varepsilon \otimes \beta_{\mathcal{A}}(I)$. We therefore have

$$\langle I \otimes \varepsilon, t \circ (\varepsilon \otimes I) \rangle = \langle I \otimes \varepsilon, \varepsilon \otimes (f^* \circ t \circ f \circ I) \rangle .$$

Apply both sides to $j \otimes k \in J \otimes J$. This yields

$$\langle j, t(k) \rangle = \langle j, (f^* \circ t \circ f)(k) \rangle ,$$

and this implies that $t = f^* \circ t \circ f$, and this finishes the proof of the lemma. □

Since $f = \beta_A = \beta_{\mathcal{A}}(I)$ is a grouplike element of J, we know that it is a coalgebra homomorphism. From the fact that $t = f^* \circ t \circ f$, it follows that f is an automorphism, its inverse being $t \circ f^* \circ t$. In the next two lemmas, we will show that f also preserves the multiplication and the antipode, and that it is therefore a Hopf algebra automorphism.

2.7. Lemma. *With notations as above, $f = \beta_A = \beta_{\mathcal{A}}(I)$ is an algebra homomorphism.*

Proof. Once again we need our 'tensoring up' construction in index $n = 2$. Consider the following grouplike elements of $G(\mathcal{J}_2)$: $\varepsilon \otimes I$, m (multiplication) and $I \otimes \varepsilon$. From Lemma 2.4, it follows that

$$\beta_{\mathcal{A}_2}(\varepsilon \otimes I) = f \circ (\varepsilon \otimes I) = \varepsilon \otimes f ,$$

$$\beta_{\mathcal{A}_2}(m) = f \circ m ,$$

$$\beta_{\mathcal{A}_2}(I \otimes \varepsilon) = f \circ (I \otimes \varepsilon) = f \otimes \varepsilon .$$

Now $(\varepsilon \otimes I) * (I \otimes \varepsilon) = m$, so $\beta_{\mathcal{A}_2}(\varepsilon \otimes I) * \beta_{\mathcal{A}_2}(I \otimes \varepsilon) = \beta_{\mathcal{A}_2}(m)$, because $\beta_{\mathcal{A}_2}$ is a group automorphism. Apply both sides to $j \otimes k \in J \otimes J$, to obtain

$$\sum_{(j),(k)} \varepsilon(j_{(1)}) f(k_{(1)}) f(j_{(2)}) \varepsilon(k_{(2)}) = f(jk) ,$$

or $f(j)f(k) = f(jk)$. $\square$

2.8. Lemma. *With notations as above, $f = \beta_A = \beta_{\mathcal{A}}(I)$ preserves the antipode.*

Proof. We know that $\beta_{\mathcal{A}}$ preserves the restriction of the antipode $S_{\mathcal{J}}$ to $G(\mathcal{J})$ (that is, it preserves the inverse of $G(\mathcal{J})$); therefore,

$$\beta_{\mathcal{A}}(S) = \beta_{\mathcal{A}}(S_{\mathcal{J}}(I)) = S_{\mathcal{J}}(\beta_{\mathcal{A}}(I)) = S \circ \beta_{\mathcal{A}}(I) = S \circ f .$$

On the other hand,

$$\beta_{\mathcal{A}}(S) = \beta_{\mathcal{A}}(I) \circ S = f \circ S ,$$

and this proves the lemma. $\square$

As announced before Lemma 2.7, we now have shown the following:

2.9. Corollary. $f = \beta_A = \beta_{\mathcal{A}}(I) \in \mathrm{Aut}_{\mathrm{Hopf}}(J)$. $\square$

2.10. Lemma. *If A and B are H-Azumaya algebras, then $\beta_{A\#B} = \beta_A \circ \beta_B$.*

Proof. Consider $\mathcal{A} = \mathrm{Hom}_R(H, A)$ and $\mathcal{B} = \mathrm{Hom}_R(H, B)$. Then

$$\beta_{A\#B} = \beta_{\mathcal{A}\#\mathcal{B}}(I) = \beta_{\mathcal{A}}(\beta_{\mathcal{B}}(I)) = \beta_{\mathcal{A}}(I) \circ \beta_{\mathcal{B}}(I) = \beta_A \circ \beta_B .$$

We used the fact that $\beta_{\mathcal{A}\#\mathcal{B}} = \beta_{\mathcal{A}} \circ \beta_{\mathcal{B}}$ (if we restrict to grouplike elements), and Lemma 2.4. Also observe that in $\mathcal{A}\#\mathcal{B}$ the smash product is taken over $\underline{J}^*$. $\square$

2.11. Lemma. *If* $[A] \in \mathrm{BD}^s(R, H)$, *then* $\beta_A = I_J$.

Proof. $[A] \in \mathrm{BD}^s(R, H)$ implies that $[\mathscr{A}] \in \mathrm{BD}^s(\underline{J}^*, \mathscr{H})$, and therefore $\beta_{\mathscr{A}}$ is the identity of $G(\mathscr{J})$, by Theorem 2.2(4). It follows that $\beta_A = \beta_{\mathscr{A}}(I) = I$. □

Lemma 2.11 also shows that β_A is independent of the choice of A in $[A]$. Therefore, we can define a homomorphism

$$\beta : \mathrm{BD}(R, H) \to \mathrm{Aut}_{\mathrm{Hopf}}(H \otimes H^*)$$

by $\beta([A]) = \beta_A$. From Lemma 2.6, it follows that the image of β is a subgroup of

$$O(R, H)_{\min} = \{f \in \mathrm{Aut}_{\mathrm{Hopf}}(H \otimes H^*) \colon q = q \circ f\}\ .$$

We therefore have a complex

$$\mathrm{BD}^s(R, H) \to \mathrm{BD}(R, H) \xrightarrow{\beta} O(R, H)_{\min}\ .$$

It would be very nice if this complex would be a short exact sequence. Let us state the following conjecture:

2.12. Conjecture. We have a short exact sequence

$$1 \to \mathrm{BD}^s(R, H) \to \mathrm{BD}(R, H) \xrightarrow{\beta} O(R, H)_{\min} \to 1\ .$$

The author knows one half of the answer to Conjecture 2.12: in Section 3, we will show that $\ker \beta = \mathrm{BD}^s(R, H)$. The remaining part is to show that the map β is surjective. Some partial results will be discussed in Section 4.

3. The kernel of the map β

The aim of this section is to show the following theorem, announced at the end of the previous section:

3.1. Theorem. *Let the map* $\beta : \mathrm{BD}(R, H) \to \mathrm{Aut}_{\mathrm{Hopf}}(H \otimes H^*)$ *be defined as in Section* 2. *Then* $\mathrm{Ker}\ \beta = \mathrm{BD}^s(R, H)$.

The proof will be a result of the following lemmas.

3.2. Lemma. *If* $[A] \in \mathrm{Ker}\ \beta$, *then* A *is* R-*central.*

Proof. From the fact that A is H-Azumaya, we know that

$$A^A = \{a \in A\colon xa = \sum_{(x)} (x_{(1)} \rightarrow a)x_{(0)},\ \text{for all } x \in A\} = R\,.$$

Take $a \in Z(A)$. If we can show that $h \rightarrow a = \varepsilon(h)a$ for all $h \in H$, then we have for all $x \in A$ that $xa = ax = \sum_{(x)} (x_{(1)} \rightarrow a)x_{(0)}$, and this will imply that $a \in R$.

We will show that $\rho(j)(a) = \varepsilon(j)a$, for all $j \in J$. Take $u \in I_{F(I)}$. Because $[A] \in \operatorname{Ker} \beta$, we know that $\beta_A = I_J$, so $\beta_{\mathscr{A}} = I_{G(\mathscr{J})}$, and therefore $S_{\mathscr{J}}(\alpha_A) = \eta_{\mathscr{J}}\varepsilon_{\mathscr{J}}$, or $\alpha_A = \eta_{\mathscr{J}}\varepsilon_{\mathscr{J}}$, and therefore $\alpha_A = \alpha_{\mathscr{A}}(I) = \eta_J \circ \varepsilon_J = (\eta_H \circ \varepsilon_J) \otimes (\eta_{H^*} \circ \varepsilon_{J^*})$. From Lemma 2.3, it follows that for all $a \in A$ and $j \in J$,

$$\sum_{(a),(j)} \langle a_{(1)}, (\eta_{H^*} \circ \varepsilon_J)(j_{(1)})\rangle u(j_{(2)})a_{(0)} = \sum_{(j)} \rho(j_{(1)})(a)u(j_{(2)})\,.$$

The left-hand side is equal to $u(j)a$, and for $a \in Z(A)$, we therefore obtain

$$au(j) = u(j)a = \sum_{(j)} \rho(j_{(1)})(a)u(j_{(2)})\,,$$

or $i(a) * u = F(I)(i(a)) * u$; so $(i(a) - F(I)(i(a))) * I_{F(I)} = 0$. Since $I_{F(I)}$ is an invertible, hence faithfully flat $\underline{J}^*$-module, it follows that $i(a) = F(I)(i(a))$, hence for all $j \in J$,

$$\varepsilon(j)a = i(a)(j) = F(I)(i(a))(j) = \sum_{(j)} \rho(j_{(1)})(i(a)(j_{(2)})) = \rho(j)(a)\,. \quad \square$$

3.3. Lemma. *If* $[A] \in \operatorname{Ker} \beta$, *and* $u \in I_{F(I)}$, *then for all* $j,k \in J$, $\rho(k)(u(j)) = \varepsilon(k)u(j)$.

Proof. It clearly suffices to prove that this property holds locally; therefore it is no restriction to assume that $I_{F(I)}$ is free, or that the action of J on A is inner. Furthermore, we know that the property holds if $j \in G(J)$ (examine the isomorphism $\mathrm{PD}(R, H) \cong \mathrm{Pic}(R) \times G(J)$ given in Proposition 1.1). The same property holds if we replace R by a faithfully flat R-algebra S. Here we will take $S = (J \otimes J)^*$. Then $S \otimes H = \mathscr{H}_2$, $S \otimes J = \mathscr{J}_2$, $S \otimes A = \mathscr{A}_2$, as in Lemma 2.6.

Suppose that the action of J on A is induced by u. By extension of scalars, the action of $\mathscr{J}_2$ on $\mathscr{A}_2$ is induced by $u_2 = I \otimes u : \mathscr{J}_2 \rightarrow \mathscr{A}_2$. We claim that u_2 is given by $u_2(\gamma) = u \circ \gamma$. Indeed, it suffices to show this for $\gamma = y^* \otimes h \in (J \otimes J)^* \otimes H$. For all $x \in J \otimes J$, we have

$$(I \otimes u)(\gamma)(x) = (y^* \otimes u(h))(x) = y^*(x)u(h) = u(y^*(x)h) = (u \circ \gamma)(x)\,,$$

proving (7). This means that for all $\gamma, \delta \in G(\mathscr{J}_2)$, $\rho(\delta)(u \circ \gamma) = u \circ \gamma$. Take $\delta = I \otimes \varepsilon$, $\gamma = \varepsilon \otimes I \in G(\mathscr{J}_2)$. Then for all $j,k \in J$.

$$(\rho(\delta)(u \circ \gamma))(j \otimes k) = \sum_{(j),(k)} \rho(j_{(1)}\varepsilon(k_{(1)}))(\varepsilon(j_{(2)})k_2) = \rho(j)u(k)\,,$$

and

$$(u \circ \gamma)(j \otimes k) = u(\varepsilon(j)k) = \varepsilon(j)u(k) .$$

The statement follows. □

3.4. Corollary. *If* $[A] \in \operatorname{Ker} \beta$, *and* $u \in I_{F(I)}$, *then, for all* $j,k \in J$, $u(j)u(k) = u(k)u(j)$.

Proof. In (3), replace a by $u(k)$. Since $\mu^* = \varepsilon$, we have

$$\sum_{(a),(j)} \langle u(k)_{(1)}, \varepsilon(j_{(1)})\rangle u(j_{(2)})u(k)_{(0)} = \sum_j \rho(j_{(1)})(u(k))(u(j_{(2)})) ,$$

or $u(j)u(k) = u(k)u(j)$. □

Recall that $\mathrm{BM}(R, H)$ is the subgroup of $\mathrm{BD}(R, H)$ consisting of classes of H-Azumaya algebras $[A]$ such that the coaction of H on A is trivial, that is, $\chi(a) = a \otimes 1$ for all $a \in A$. Consider the restriction

$$\beta|_{\mathrm{BM}} : \mathrm{BM}(R, H) \to \mathrm{Aut}_{\mathrm{Hopf}}(H \otimes H^*) .$$

We denote $\mathrm{BM}^s(R, H) = \mathrm{BM}(R, H)$. Recall the following theorem, due to Beattie [5]:

3.5. Theorem. $\mathrm{BM}(R, H) \cong \mathrm{Gal}(R, H) \times \mathrm{Br}(R)$. □

3.6. Lemma. $\operatorname{Ker}(\beta|_{\mathrm{BM}}) = \mathrm{BM}^s(R, H)$.

Proof. Let $[A] \in \operatorname{Ker}(\beta|_{\mathrm{BM}})$. Then we know that there exists a faithfully flat R-algebra S (a Zariski covering will do) such that $S \otimes H$ acts innerly on $S \otimes A$.

If $[S \otimes A] \in \mathrm{BM}^s(S, S \otimes H)$, then $[A] \in \mathrm{BM}^s(R, H)$, and therefore we may assume that the action of H on A is inner. In this case, we can explicitly describe the map $S : \mathrm{BM}(R, H) \to \mathrm{Gal}(R, H)$. It is defined as follows (we refer to [9] for details): suppose that the action of H on A is induced by the map $u : H \to A$, with convolution inverse v, then

$$S(A) = \{v(h_{(1)})\#h_{(2)} : h \in H\} \subset A\#H .$$

It suffices to show that $S(A)$ may be split in $\mathrm{Gal}(R, H)$, or, equivalently, that $S(A)$ is commutative (cf. [10]).

Using Lemma 3.3 and Corollary 3.4, we have, for all $h,k \in H$,

$$\begin{aligned}
&\Big(\sum_{(h)} v(h_{(1)})\#h_{(2)}\Big)\Big(\sum_{(k)} v(k_{(1)})\#k_{(2)}\Big)\\
&\quad=\sum_{(h),(k)} v(h_{(1)})(h_{(2)}\to v(k_{(2)}))\#h_{(3)}k_{(2)}\\
&\quad=\sum_{(h),(k)} v(h_{(1)})v(k_{(1)})\#h_{(2)}k_{(2)} \qquad \text{(by Lemma 3.3)}\\
&\quad=\sum_{(h),(k)} v(k_{(1)})v(h_{(1)})\#k_{(2)}h_{(2)} \qquad \text{(by Corollary 3.4)}\\
&\quad=\Big(\sum_{(k)} v(k_{(1)})\#k_{(2)}\Big)\Big(\sum_{(h)} v(h_{(1)})\#h_{(2)}\Big),
\end{aligned}$$

and this proves the lemma. □

3.7. Lemma. *Suppose* $[A]\in \operatorname{Ker}\beta$. *Then* A *is an Azumaya algebra.*

Proof. It suffices to show that $A\otimes S$ is Azumaya for some faithfully flat R-algebra S. Therefore, replacing R by S, we may assume that the action of H on A is inner, and induced by $u: H\to A$, with convolution inverse v. Define $f: H\otimes H\to R$ by

$$f(h\otimes k)=\sum_{(h),(k)} v(h_{(2)})v(k_{(2)})u(h_{(1)}k_{(1)}) .$$

From Lemma 3.4, if follows that f is symmetric, $(f(h\otimes k)=f(k\otimes h))$. It is straightforward to show that f is a normalized Sweedler 2-cocycle (cf. [32] for a discussion of Sweedler's cohomology). On A, we define a new multiplication as follows:

$$m(a\otimes b)=\sum_{(a),(b)} f(a_{(1)}\otimes b_{(1)})b_{(2)}a_{(2)} .$$

A furnished by this new multiplication is denoted by $A^{f\text{-opp}}$. Define $j: A\#\bar{A}\to A\otimes A^{f\text{-opp}}$ by $j(a\#b)=\sum_{(b)} au(b_{(1)})\otimes b_{(0)}$. We claim that j is an R-algebra homomorphism. Indeed, on one hand we have that

$$\begin{aligned}
j((a\#b)(c\#d)) &= \sum_{(b)} j(a(b_{(1)}\to c)\#b_{(0)}d)\\
&=\sum_{(b)} j(au(b_{(1)})cv(b_{(2)})\#b_{(0)}d)\\
&=\sum_{(b),(d)} au(b_{(1)})cv(b_{(2)})u(b_{(3)}d_{(1)})\otimes b_{(0)}d_{(0)}\\
&=\sum_{(b),(d)} au(b_{(1)})cu(d_{(1)})f(b_{(2)}\otimes d_{(2)})\otimes b_{(0)}d_{(0)} ;
\end{aligned}$$

on the other hand,

$$j(a\#b)j(c\#d) = \sum_{(b),(d)} (au(b_{(1)}) \otimes b_{(0)})(cu(d_{(1)}) \otimes d_{(0)})$$
$$= \sum_{(b),(d)} au(b_{(1)})cu(d_{(1)}) \otimes f(b_{(2)} \otimes d_{(2)})b_{(0)}d_{(0)} .$$

j is invertible; its inverse is given by $j^{-1}(a \otimes b) = \sum_{(b)} av(b_{(1)})\#b_{(0)}$. Since f is a symmetric cocycle, there exists a faithfully flat R-algebra S such that $f \otimes 1_S$ is a coboundary. Replacing R by S, we obtain that $A^{f\text{-opp}} \cong A^{\text{opp}}$. But then we have

$$A \otimes A^{\text{opp}} \cong A \otimes A^{f\text{-opp}} \cong A\#\bar{A} \cong \text{End}_R(A) ,$$

and this proves that A is Azumaya. $\square$

Proof of Theorem 3.1. Take $[A] \in \text{Ker}\,\beta$. By Lemma 3.7, A is Azumaya. The map ρ defined in 2.1 defines a J-action on A, so A is a J-module Azumaya algebra. From Lemma 3.6, it follows that $[A] \in \text{BM}^{\text{s}}(R, J)$. But this implies that $[A] \in \text{BD}^{\text{s}}(R, H)$. $\square$

The result of Sections 2 and 3 may be summarized in the following theorem:

3.8. Theorem. *We have an exact sequence*

$$1 \to \text{BD}^{\text{s}}(R, H) \to \text{BD}(R, H) \xrightarrow{\beta} O(R, H)_{\min} ,$$

where $O(R, H)_{\min} = \{f \in \text{Aut}_{\text{Hopf}}(H \otimes H^*)\colon q \circ f = q\}$. $\square$

It is an open problem whether the map β is surjective in general. For a similar problem concerning Galois objects, we refer to [13]. In [14], Beattie and the author have shown that β is surjective under the following conditions: $H = RG$, with G is a finite abelian group, $|G|$ is invertible in R, and R contains a primitive $\exp(G)$-th root of unity.

4. Deegan's subgroup of the Brauer–Long group

Remark that we have an embedding $i : \text{Aut}_{\text{Hopf}}(H) \to O(R, H)_{\min}$, given by

$$i(f) = f \otimes (f^{-1})^* , \quad \text{or} \quad i(f)(h \otimes h^*) = f(h) \otimes (f^{-1})^*(h^*) .$$

Indeed, for all $h \otimes h^* \in J$, we have that

$$(q \circ i(f))(h \otimes h^*)$$
$$= \langle f(h), (f^{-1})^*(h^*)\rangle = \langle (f^{-1} \circ f)(h), h^* \rangle = q(h \otimes h^*) .$$

In this section, we will show that $i(\mathrm{Aut}_{\mathrm{Hopf}}(H))$ is a subgroup of Im β. In the case where $H = RG$ is a groupring, it is known that $\mathrm{Aut}(G) = \mathrm{Aut}_{\mathrm{Hopf}}(RG)$ is a subgroup of $\mathrm{BD}(R, G)$; this was shown by Deegan in [19]: he defined a subgroup $\mathrm{BT}(R, G)$ of $\mathrm{BD}(R, G)$, and then he proved that $\mathrm{BT}(R, G) \cong \mathrm{Aut}(G)$. We will generalize Deegan's construction to the general case where H is a Hopf algebra. Let $\mathrm{BT}(R, H)$ be the subset of $\mathrm{BD}(R, H)$ consisting of classes of algebras represented by an R-algebra of the form $A = \mathrm{End}_R(M)$, where M is at once a faithfully projective H-module and H-comodule (but not an H-dimodule), and where the action and coaction of H on A are induced by the action and coaction of H on M. The results of this section are summarized in the following theorem; the proof will follow from Lemmas 4.2–4.8.

4.1. Theorem. $\mathrm{BT}(R, H)$ *is a subgroup of* $\mathrm{BD}(R, H)$. *The map*

$$\beta : \mathrm{BD}(R, H) \to O(R, H)_{\min}$$

restricts to an isomorphism.

$$\beta_t : \mathrm{BT}(R, H) \to i(\mathrm{Aut}_{\mathrm{Hopf}}(H)) .$$

Consequently $i(\mathrm{Aut}_{\mathrm{Hopf}}(H)) \subset \mathrm{Im}\ \beta$.

4.2. Lemma. *If* $[A] \in \mathrm{BT}(R, H)$, *then* $\beta_A = f \otimes (f^*)^{-1}$, *for some* $f \in \mathrm{Aut}_{\mathrm{Hopf}}(H)$.

Proof. Recall the definition of $\alpha_{\mathscr{A}}$ (cf. Theorem 2.2):

$$\alpha_{\mathscr{A}} : G(J) \xrightarrow{F} \mathscr{H}\text{-}\mathrm{Aut}(\mathscr{A}) \to \mathrm{PD}(\underline{J}^*, \mathscr{H}) \to G(\mathscr{J}) .$$

We need to find $\alpha_{\mathscr{A}}(I_J)$. Observe that $I_J = (I_H \otimes \varepsilon_{H^*}) * (\varepsilon_H \otimes I_{H^*})$; we will first compute $\alpha_{\mathscr{A}}(I_H \otimes \varepsilon_{H^*})$. Define $u \in \mathscr{A} = \mathrm{Hom}_R(J, A)$ by $u(h \otimes h^*)(m) = \langle h^*, 1\rangle (h \rightharpoonup m)$, for all $h \in H, h^* \in H^*, m \in M$.

We then have for all $\mu \in \mathscr{H} = \mathrm{Hom}_R(J, H)$ that

$$\mu \rightharpoonup u = (\varepsilon_{\mathscr{H}^*} \circ \mu) * u . \tag{8}$$

Indeed, for all $j = k \otimes k^* \in J$ and for all $m \in M$, we have that

$$\begin{aligned}
&(\mu \rightharpoonup u)(k \otimes k^*)(m) \\
&\quad = \sum_{(k),(k^*)} (\mu(k_{(1)} \otimes k^*_{(1)}) \rightharpoonup u(k_{(2)} \otimes k^*_{(2)}))(m) \\
&\quad = \sum_{(k),(k^*),\mu(k_{(1)}\otimes k^*_{(1)})} \mu(k_{(1)} \otimes k^*_{(1)})_{(1)} \\
&\qquad\qquad \rightharpoonup u(k_{(2)} \otimes k^*_{(2)})(S(\mu(k_{(1)} \otimes k^*_{(1)})_{(2)}) \rightharpoonup m)
\end{aligned}$$

$$= \sum_{(k),(k^*),\mu(k_{(1)}\otimes k^*_{(1)})} \langle k^*_{(2)}, 1\rangle \mu(k_{(1)}\otimes k^*_{(1)})_{(1)}$$
$$\to (k_{(2)} \to (S(\mu(k_{(1)}\otimes k^*_{(1)})_{(2)}) \to m))$$
$$= \sum_{(k),(k^*),\mu(k_{(1)}\otimes k^*_{(1)})} \langle k^*_{(2)}, 1\rangle$$
$$\times \{(\mu(k_{(1)}\otimes k^*_{(1)})_{(1)} k_{(2)} S(\mu(k_{(1)}\otimes k^*_{(1)})_{(2)})\} \to m$$
$$= \sum_{(k),(k^*)} \varepsilon(\mu(k_{(1)}\otimes k^*_{(1)}))\langle k^*_{(2)}, 1\rangle (k_{(2)} \to m)$$
$$= (\varepsilon_{\mathscr{H}^*}(\mu) * u)(m)\,.$$

$$u \in I_{F(I_H\otimes\varepsilon_{H^*})}\,. \tag{9}$$

Invoking Lemma 2.3, it suffices to show that, for all $k\otimes k^* \in J$ and $a \in A$;

$$u(k\otimes k^*)a = \sum_{(k),(k^*)} \rho(k_{(1)}\otimes \varepsilon(k^*_{(1)}))(a)u(k_{(2)}\otimes k^*_{(2)})\,,$$

or

$$u(k\otimes k^*)a = \sum_{(k)} (k_{(1)} \to a)u(k_{(2)}\otimes k^*)\,.$$

The left-hand side evaluated at $m \in M$ gives $\varepsilon(k^*)(k \to a(m))$, while the right-hand side is $\sum_{(k)} k_{(1)} \to a(S(k_{(2)}) \to (\varepsilon(k^*)k_{(3)} \to m))$; it is clear that both sides are equal.

From (8) and (9), it follows that $\alpha_{\mathscr{A}}(I_H\otimes\varepsilon_{H^*}) = \lambda\otimes\varepsilon_{\mathscr{H}^*}$, for some $\lambda\in G(\mathscr{H})$. From a duality argument, it then follows that $\alpha_{\mathscr{A}}(\varepsilon_H\otimes I_{H^*}) = \varepsilon_{\mathscr{H}}\otimes\lambda'$, for some $\lambda'\in G(\mathscr{H}^*)$. Therefore,

$$\alpha_A = \alpha_{\mathscr{A}}(I) = \alpha_{\mathscr{A}}(I_H\otimes\varepsilon_{H^*})\alpha_{\mathscr{A}}(\varepsilon_H\otimes I_{H^*}) = \lambda\otimes\lambda'\,,$$

and $\beta_A = f\otimes f'$. From the fact that β_A is a Hopf algebra automorphism, it follows that f and f' are automorphisms. $\beta_A \in O(R,H)_{\min}$ implies that $f' = (f^*)^{-1}$. □

4.3. Lemma. BT(R, H) *is a multiplicative subset of* BD(R, H).

Proof. Our proof is a straightforward generalization of [19, Theorem 2.6].

Take $A = \mathrm{End}_R(M) = M^*\otimes M$, $B = \mathrm{End}_R(N) = N^*\otimes N$, representing elements of BT(R, H). We have to show that $[A\#B] \in$ BT(R, H). In the sequel, we will identify

$$A\otimes B = \mathrm{End}_R(M)\otimes\mathrm{End}_R(N) = \mathrm{End}_R(M\otimes N)\,.$$

As we have seen in Lemma 4.2, the action of H on A resp. B is inner, and is induced by $u : H \to A$ resp. $v : H \to B$, defined by $u(h)(m) = h \to m$ and $v(h)(n) = h \to n$, for all $m \in M$, $n \in N$. Remark that u and v are multiplicative maps, and that H acts trivially on u and v: $h \to u(k) = \varepsilon(h)u(k)$ and $h \to v(k) = \varepsilon(h)v(k)$, for all $h,k \in H$.

From [28, Proposition 3.7], it follows that we have an H-module algebra isomorphism

$$\Phi : A\#B \to A \otimes B \cong \mathrm{End}_R(M \otimes N)\ ,$$

given by

$$\Phi(a\#b) = \sum_{(b)} au(b_{(1)})\#b_{(0)}\,.$$

It is easily verified that the inverse of the map Φ is defined by

$$\Phi^{-1}(a\#b) = \sum_{(b)} au(S(b_{(1)}))\#b_{(0)}\,.$$

Note that Long's part of the proof where he shows that F is also an H-comodule map does not hold here, because M is not an H-dimodule (this was already observed by Deegan in [19]). We define $\mathrm{End}_R(M \otimes N)_{(1)}$ to be $\mathrm{End}_R(M \otimes N)$, with the H-dimodule algebra structure induced by Φ (then the H-module algebra structure is induced by the one on $M \otimes N$). Then

$$\Phi : A\#B \to \mathrm{End}_R(M \otimes N)_{(1)}$$

is an H-dimodule algebra isomorphism.

In a similar way, it follows from [28, Proposition 3.10] that we have an H-comodule algebra isomorphism

$$\rho : A\#B \to A \otimes B \cong \mathrm{End}_R(M \otimes N)\ ,$$

given by

$$\rho(a\#b) = \sum_{(n)} \left(\sum (n_{(1)}) \to a\right) \otimes (n^* \otimes n_{(0)})\ ,$$

for all $a \in A$, $b = n^* \otimes n \in B \cong N^* \otimes N$.

We define $\mathrm{End}_R(M \otimes N)_{(2)}$ to be $\mathrm{End}_R(M \otimes N)$ with H-dimodule algebra structure induced by the map ρ. Then $\rho : A\#B \to \mathrm{End}_R(M \otimes N)_{(2)}$ is an H-dimodule algebra isomorphism. Now consider the H-dimodule algebra isomorphisms

$$\mathrm{End}_R(M\otimes N)_{(1)} \xrightarrow{\Phi^{-1}} A\otimes B \xrightarrow{\rho} \mathrm{End}_R(M\otimes N)_{(2)} .$$

The coaction of H on $\mathrm{End}_R(M\otimes N)_{(2)}$ is induced by the coaction of H on $M\otimes N$. If we can show that the action of H on $\mathrm{End}_R(M\otimes N)_{(2)}$ is induced by an action on $M\otimes N$, then we are done. Now the action of H on $\mathrm{End}_R(M\otimes N)_{(1)}$ is induced by the action of H on $M\otimes N$; therefore it is induced by the map

$$\Gamma = (u\otimes v)\circ\Delta : H \to \mathrm{End}_R(M\otimes N)_{(1)} ,$$

given by

$$\Gamma(h) = \sum_{(h)} u(h_{(1)})\otimes v(h_{(2)}) .$$

The action of H on $\mathrm{End}_R(M\otimes N)_{(2)}$ is therefore induced by $\Omega = \rho\circ\Phi^{-1}\circ\Gamma$; using the fact that H acts trivially on u, we obtain that

$$\Omega(h) = \sum_{(h),v(h_{(2)})} u(h_{(1)}S(v(h_{(2)})_{(1)}))\otimes v(h_{(2)})_{(0)} .$$

Define an action $\rightsquigarrow$ of H on $M\otimes N$ by $h\rightsquigarrow(m\otimes n) = \Omega(h)(m\otimes n)$. From the fact that u,v preserve multiplication, it follows that Γ, and therefore Ω preserves the multiplication. But this means that the action defined above makes $M\otimes N$ into an H-module. Moreover, this action induces the action of H on $\mathrm{End}_R(M\otimes N)_{(2)}$, and this finishes the proof of our lemma. □

The foregoing lemma shows that β restricts to a homomorphism

$$\beta_t : \mathrm{BT}(R, H) \to i(\mathrm{Aut}_{\mathrm{Hopf}}(H));$$

let us prove that β_t is surjective.

For $f\in\mathrm{Aut}_{\mathrm{Hopf}}(H)$, we define $g\in\mathrm{Hopf}(H, H)$ by $g = I*(S\circ f)$, or

$$g(h) = \sum_{(h)} h_{(1)}S(f(h_{(2)})) = \sum_{(h)} h_{(1)}f(S(h_{(2)})) .$$

Let H_f be equal to H as an *R-module*. On H_f, we define an H-action and an H-coaction as follows:

$$\chi(h) = \Delta(h) = \sum_{(h)} h_{(1)}\otimes h_{(2)} , \qquad k\to h = g(k)h ,$$

for all $h\in Hf$, $k\in H$. Remark that H_f, although it is an H-module and an H-comodule, is not an H-dimodule, since the action and coaction are not compatible (this may be seen easily in the case where $H = RG$); we leave details to the reader.

4.4. Lemma. $A_f = \mathrm{End}_R(H_f)$, *with the induced action and coaction, is an H-dimodule algebra.*

Proof. Because H is finitely generated projective, we can identify $H_f^* \otimes H_f$ and $\mathrm{End}_R(H_f)$ by the isomorphism Φ, given by $\Phi(\varphi \otimes h)(l) = \varphi(l)(h)$ ($\varphi \in H_f^*$, $h, l \in H_f$). We first describe the action and coaction of H on $H_f^* = \mathrm{Hom}_R(H, R)$. The action is given by

$$(k \rightharpoonup \varphi)(h) = \varphi(S(k) \rightharpoonup h) = \varphi(S(g(k))h)\,.$$

The coaction $\chi : H_f^* \to H_f^* \otimes H \cong \mathrm{Hom}_R(H_f, H)$ is given by

$$\chi(\varphi)(h) = \sum_{(h)} \varphi(h_{(1)})S(h_{(2)}),$$

that means that if we write

$$\chi(\varphi) = \sum_{(j)} \varphi_{(0)} \otimes \varphi_{(1)} \in H_f^* \otimes H\,,$$

then, for all $h \in H$,

$$\sum_{(h)} \varphi(h_{(1)})S(h_{(2)}) = \sum_{(j)} \varphi_{(0)}(h)\varphi_{(1)}\,. \tag{10}$$

On the tensor product $H_f^* \otimes H_f$, action and coaction are given by

$$k \rightharpoonup (\varphi \otimes h) = \sum_{(k)} (k_{(1)} \rightharpoonup \varphi) \otimes (k_{(2)} \rightharpoonup h)\,,$$

$$\chi(\varphi \otimes h) = \sum_{(\varphi),(h)} (\varphi_{(0)} \otimes h_{(1)}) \otimes j_{(1)}h_{(2)}\,,$$

for all $k \in H$, $h \in H_f$, $\varphi \in H_f^*$.

In order to show the lemma, we have to verify that

$$\chi \circ \psi = (\psi \otimes 1) \circ (1 \otimes \chi)\,,$$

or that the following diagram is commutative:

$$\begin{tikzcd}
H \otimes (H_f^* \otimes H_f) \arrow[r, "\psi"] \arrow[d, "1\otimes\chi"] & H_f^* \otimes H_f \arrow[d, "\chi"] \\
H \otimes (H_f^* \otimes H_f) \otimes H \arrow[r, "\psi\otimes 1"'] & (H_f^* \otimes H_f) \otimes H \cong \mathrm{Hom}(H_f, H_f \otimes H)
\end{tikzcd}$$

Take $k \otimes \varphi \otimes h \in H \otimes H_f^* \otimes H_f$. Then

$$
\begin{aligned}
&(\chi \circ \psi)(k \otimes \varphi \otimes h) \\
&= \chi\Big(\sum_{(k)} (k_{(1)} \rightarrow \varphi) \otimes (k_{(2)} \rightarrow h)\Big) \\
&= \chi\Big(\sum_{(k)} (k_{(1)} \rightarrow \varphi) \otimes g(k_{(2)})h\Big) \\
&= \sum_{(k),(k_{(1)}\rightarrow\varphi),(h)} (k_{(1)} \rightarrow \varphi)_{(0)} \otimes g(k_{(2)}h_{(1)}) \otimes (k_{(1)} \rightarrow \varphi)_{(1)} g(k_{(3)})h_{(2)} .
\end{aligned}
$$

If we consider this identity in $\mathrm{Hom}_R(H_f, H_f \otimes H)$, and if we apply it to $l \in H_f$, then we obtain

$$
\begin{aligned}
&(\chi \circ \psi)(k \otimes \varphi \otimes h)(l) \\
&= \sum_{(k),(k_{(1)}\rightarrow\varphi),(h)} g(k_{(2)}h_{(1)}) \otimes \langle (k_{(1)} \rightarrow \varphi)_{(0)}, l \rangle (k_{(1)} \rightarrow \varphi)_{(1)} g(k_{(3)})h_{(2)} \\
&= \sum_{(k),(l),(h)} g(k_{(2)}h_{(1)}) \otimes \langle (k_{(1)} \rightarrow \varphi), l_{(1)} \rangle S(l_{(2)}) g(k_{(3)})h_{(2)} \\
&= \sum_{(k),(l),(h)} g(k_{(2)}h_{(1)}) \otimes \Big\langle \varphi, \sum (g(k_{(1)}))l_{(2)} \Big\rangle S(l_{(2)}) g(k_{(3)})h_{(2)} .
\end{aligned}
$$

We also have

$$
\begin{aligned}
&(\psi \otimes 1)(1 \otimes \chi)(k \otimes \varphi \otimes h) \\
&= (\psi \otimes 1)\Big(\sum_{(\varphi),(h)} k \otimes \varphi_{(0)} \otimes h_{(1)} \otimes \varphi_{(1)}h_{(2)}\Big) \\
&= \sum_{(\varphi),(h),(k)} (k_{(1)} \rightarrow \varphi_{(0)}) \otimes (k_{(2)} \rightarrow h_{(1)}) \otimes \varphi_{(1)}h_{(2)} ,
\end{aligned}
$$

so, for all $l \in H_f$,

$$
\begin{aligned}
&(\psi \otimes 1)(1 \otimes \chi)(k \otimes \varphi \otimes h)(l) \\
&= \sum_{(\varphi),(h),(k)} g(k_{(2)})h_{(1)} \otimes (k_{(1)} \rightarrow \varphi_{(0)})(l) \varphi_{(1)}h_{(2)} \\
&= \sum_{(\varphi),(h),(k)} g(k_{(2)})h_{(1)} \otimes \varphi_{(0)}(S(g(k_{(1)}))l) \varphi_{(1)}h_{(2)} \\
&= \sum_{(\varphi),(h),(k)} g(k_{(2)})h_{(1)} \otimes \varphi(S(g(k_{(2)}))l_{(1)}) g(k_{(3)}) S(l_{(2)}) h_{(2)} ,
\end{aligned}
$$

by (10). This finishes the proof of Lemma 4.4. □

For $h \in H$, we define $x(h) \in A_f = \mathrm{End}_R(H_f)$ by

$$x(h)(k) = h \rightarrow k = g(h)k .$$

Observe that this enables us to rewrite the action of H on A_f under the following form:

$$h \rightarrow a = \sum_{(h)} x(h_{(1)}) a x(S(h_{(2)})) ,$$

for $h \in H$, $a \in A_f$.

4.5. Lemma. $\chi(x(h)) = \sum_{(h)} x(h_{(0)}) \otimes g(h_{(1)})$.

Proof. For $f \in \mathrm{End}_R(H_f)$, $\chi(f) \in \mathrm{Hom}_R(H_f, H_f \otimes H)$ is given by

$$\chi(f)(k) = \sum_{(k), f(k_{(0)})} f(k_{(0)})_{(0)} \otimes f(k_{(0)})_{(1)} S(k_{(1)});$$

if we replace f by $x(h)$, then we obtain

$$\begin{aligned}
\chi(h)(k) &= \sum_{(k),(g(h)k_{(0)})} (g(h)k_{(0)})_{(0)} \otimes (g(h)k_{(0)})_{(1)} S(k_{(1)}) \\
&= \sum_{(h),(k)} g(h_{(0)})k_{(0)} \otimes g(h_{(1)})k_{(1)} S(k_{(2)}) \\
&= \sum_{(h)} g(h_{(0)})k \otimes g(h_{(1)}) \\
&= \Big\{ \sum_{(h)} x(h_{(0)}) \otimes g(h_{(1)}) \Big\}(k) .
\end{aligned}$$

4.6. Lemma. *A_f is an H-Azumaya algebra.*

Proof. We have to show that the maps F and G defined in 1.2 are isomorphisms. We will restrict ourselves to showing that $F : A_f \# \bar{A}_f \rightarrow \mathrm{End}_R(A_f)$ is surjective. A count of ranks then shows that F is also injective. A similar computation will show that G is an isomorphism.

Recall that

$$F(a\#b)(c) = \sum_{(b)} a(b_{(1)} \rightarrow c) b_{(0)} .$$

Being an endomorphism ring, A_f is an Azumaya algebra. Therefore the map

$$F' : A_f \otimes A_f^{\mathrm{opp}} \rightarrow \mathrm{End}_R(A_f) ,$$

given by $F'(a\otimes b)(c)=acb$ is an isomorphism. Take $\alpha\in \mathrm{End}_R(A_f)$, and suppose that $\alpha=F'(\sum_i a_i\otimes b_i)$, i.e. for all $c\in A_f$, $\alpha(c)=\sum_i a_icb_i$. In $A_f\#\bar{A}_f$, we consider the element

$$\Gamma=\sum_{i,(b_i)} a_ix(S(f^{-1}(b_{i_{(2)}})))\otimes x(f^{-1}(b_{i_{(1)}}))b_{i_{(0)}}\ .$$

We have that

$$\begin{aligned}
&\chi\Big(\sum_{(b_i)} x(f^{-1}(b_{i_{(1)}}))b_{i_{(0)}}\Big)\\
&\quad=\sum_{(b_i)} x(f^{-1}(b_{i_{(1)}}))b_{i_{(0)}}\otimes g(f^{-1}(b_{i_{(2)}}))b_{i_{(3)}}\\
&\quad=\sum_{(b_i)} x(f^{-1}(b_{i_{(1)}}))b_{i_{(0)}}\otimes f^{-1}(b_{i_{(2)}})S(f(f^{-1}(b_{i_{(3)}})))b_{i_{(4)}}\\
&\quad=\sum_{(b_i)} x(f^{-1}(b_{i_{(1)}}))b_{i_{(0)}}\otimes f^{-1}(b_{i_{(2)}});
\end{aligned}$$

therefore

$$\begin{aligned}
F(\Gamma)(c)&=\sum_{i,(b_i)} a_ix(S(f^{-1}(b_{i_{(1)}})))(f^{-1}(b_{i_{(2)}})\rightarrow c)x(f^{-1}(b_{i_{(3)}}))b_{i_{(0)}}\\
&=\sum_{i,(b_i)} a_ix(S(f^{-1}(b_{i_{(1)}})))x(f^{-1}(b_{i_{(2)}}))cx(S(f^{-1}(b_{i_{(3)}})))\\
&\qquad\times x(f^{-1}(b_{i_{(4)}}))b_{i_{(0)}}\\
&=\sum_i a_icb_i=\alpha(c)\ ,
\end{aligned}$$

finishing the proof of the lemma. □

4.7. Lemma. $\beta_{\mathcal{A}_f}=f\otimes(f^*)^{-1}$.

Proof. It is clear that $[A_f]\in \mathrm{BT}(R,H)$, so therefore we may use Lemma 4.2. Looking at the proof of Lemma 4.2, it follows that it suffices to show that

$$\chi_{\mathcal{A}}(u)=u\otimes(g\otimes\eta_H\circ\varepsilon_{H^*})\ ,$$

where $u\in\mathcal{A}_f=\mathrm{Hom}_R(J,A_f)$ is defined by

$$u(k\otimes k^*)(h)=\langle k^*,1\rangle(k\rightarrow h)=\langle k^*,1\rangle g(k)h\ .$$

Observe first that $\chi_{\mathcal{A}}(u)\in\mathcal{A}\otimes\mathcal{H}=\mathrm{Hom}_R(J,A\otimes H)$ is given by

$$\chi_{\mathscr{A}}(u) = \chi_A \circ u \ .$$

Take $j = k \otimes k^* \in J$ and $h \in H$. Then

$$\begin{aligned}
&\chi_{\mathscr{A}}(u)(k \otimes k^*)(h) \\
&\quad = \chi_A(u(k \otimes k^*))(h) \\
&\quad = \sum_{(h),(u(j)(h))} (u(j)(h))_{(0)} \otimes (u(j)(h))_{(1)} S(h_{(2)}) \\
&\quad = \sum_{(h),(k)} \langle k^*, 1 \rangle \, g(k_{(1)}) h_{(1)} \otimes g(k_{(2)}) h_{(2)} S(h_{(3)}) \\
&\quad = \sum_{(k)} \langle k^*, 1 \rangle \, g(k_{(1)}) h \otimes g(k_{(2)}) \ .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&(u \otimes (g \otimes \eta_H \varepsilon_{H^*}))(k \otimes k^*)(h) \\
&\quad = \sum_{(k),(k^*)} u(k_{(1)} \otimes k^*_{(1)}) \otimes (g(k_{(2)}) \otimes \eta_H(\varepsilon_{H^*}(k^*_{(2)})))(h) \\
&\quad = \sum_{(k)} g(k_{(1)}) \langle k^*, 1 \rangle h \otimes g(k_{(2)}) \ . \qquad \square
\end{aligned}$$

The proof of Theorem 4.1 will be finished after the proof of the following lemma:

4.8. Lemma. $\beta_t : \mathrm{BT}(R, H) \to i(\mathrm{Aut}_{\mathrm{Hopf}}(H))$ *is injective.*

Proof. Take $[A] = [\mathrm{End}_R(M)] \in \mathrm{BT}(R, H)$, and suppose that $\beta_t([A]) = I_J$, or $\alpha_A = \varepsilon_J$. We will show that this implies that M is an H-dimodule, that is,

$$\sum_{(h \to m)} (h \to m)_{(0)} \otimes (h \to m)_{(1)} = \sum_{(m)} (h \to m_{(0)}) \otimes m_{(1)} \ , \tag{11}$$

for all $h \in H$, $m \in M$. From the previous lemmas, we know that the action of H on A is induced by the map $x : H \to A$, given by $x(h)(m) = h \to m$, that is, $x(h) = u(h \otimes 1)$, where u is defined as in Lemma 4.3. Now $\chi_{\mathscr{A}}(u) = \chi_A \circ u = u \circ \varepsilon_J$, since $\alpha_A = \varepsilon_J$. Therefore,

$$\chi_A \circ x = u \otimes \varepsilon_H \quad \text{or} \quad \chi_A(x(h)) = \sum_{(h)} x(h_{(1)}) \otimes \varepsilon(h_{(2)}) = x(h) \otimes 1 \ .$$

Recall that

$$\chi_A : A \to A \otimes H = \mathrm{End}_R(M) \otimes H = \mathrm{Hom}_R(M, M \otimes H) \ ,$$

so

$$\chi_A(x(h))(m) = (h \rightarrow m) \otimes 1 . \tag{12}$$

On the other hand, for all $a \in A$, $\chi_A(a)$ is defined by

$$\chi_A(a)(m) = \sum_{(m),(a(m)_{(0)})} a(m_{(0)})_{(0)} \otimes a(m_{(0)})_{(1)} S(m_{(1)}) .$$

Take $a = x(h)$; then we obtain

$$\chi_A(x(h))(m) = \sum_{(m),(a(m)_{(0)})} (h \rightarrow m_{(0)})_{(0)} \otimes (h \rightarrow m_{(0)})_{(1)} S(m_{(1)}) . \tag{13}$$

(11) now follows from (12) and (13). □

5. Hopf algebraic version of the 'minus' algebra

In the classical theory of the Brauer–Wall group over a field k (cf. [36]), it has been known for a long time that there exists basically two types of 'Brauer–Wall–Azumaya algebras' (AZUMAYA algebras in the terminology of Wall): the ones which are R-central (called 'plus'-algebras), and the ones which are noncentral (called 'minus'-algebras). The 'plus'-algebras form a subgroup $\mathrm{BW}^+(R)$ of the Brauer–Wall group $\mathrm{BW}(R)$, and, if $\mathrm{char}(k) \neq 2$, then the quotient group $\mathrm{BW}(R)/\mathrm{BW}^+(R)$ is the cyclic group of order two. Its nontrivial element is represented by the groupring kC_2. For the Brauer–Long group $\mathrm{BD}(R, G)$, we have a similar phenomenon, if G is cyclic. If $G = C_n$, and if n is invertible in R, and R contains a primitive n-th root of unity, then the subset of $\mathrm{BD}(R, G)$ consisting of classes of H-Azumaya algebras which are central forms a subgroup of index two. The nontrivial element of the quotient group is represented by RG with a suitable action. For details, we refer to [6, 8, 20, 29].

If G is not cyclic, then the situation is more complicated, as was first noted by Orzech in [29]; however, under certain conditions, we still have a type of 'minus'-algebra, that is, we can give RG a suitable G-action such that it becomes a G-Azumaya algebra. In this section, we generalize this construction to the case of a Hopf algebra: we show that H can be made into an H-Azumaya algebra if there exists a selfdual Hopf algebra isomorphism $f : H \rightarrow H^*$.

Suppose that $f : H \rightarrow H^*$ is a selfdual Hopf algebra isomorphism, i.e. $\langle f(h), k \rangle = \langle h, f(k) \rangle$, for all $h, k \in H$. We may then define a Hopf algebra isomorphism λ of J by $\lambda(h \otimes h^*) = f^{-1}(h^*) \otimes f(h)$, that is, $\lambda = (f^{-1} \otimes f) \circ t$. It is clear that $\lambda \in O(R, H)_{\min}$, since

$$q(\lambda(h \otimes h^*)) = \langle f^{-1}(h^*), f(h) \rangle = \langle h^*, h \rangle = q(h \otimes h^*) ,$$

for all $h \otimes h^* \in J$.

5.1. Theorem. *If $f : H \to H^*$ is a Hopf algebra isomorphism such that $f = f^*$, then*

$$\lambda = (f^{-1} \otimes f) \circ t \in \operatorname{Im} \beta .$$

Proof. It suffices to construct an inverse image $[A]$ in $\mathrm{BD}(R, H)$ for λ. Let $A = H$ as an R-algebra, with H-coaction defined by $\chi_A = \Delta$, and H-action given by

$$h \rightarrow k = \sum_{(k)} \langle f(S(h)), k_{(2)} \rangle k_{(1)} ,$$

for all $h, k \in H$. In Lemma 5.2, we will show that A is H-Azumaya, and in Lemma 5.3 that $\beta_A = \lambda$. This will prove Theorem 5.1.

5.2. Lemma. *With notations as above, A is an H-Azumaya algebra.*

Proof. It is clear that A is faithfully projective as an R-module. Consider

$$F : A \# \bar{A} \to \operatorname{End}_R(A) \cong H^* \otimes H .$$

Recall from 1.2 that F is defined by

$$F(h \# l)(k) = \sum_{(b)} h(l_{(1)} \rightarrow k) l_{(0)} = \sum_{(k),(l)} \langle f(S(l_{(1)})), k_{(1)} \rangle h k_{(2)} l_{(2)} .$$

In order to prove that F is surjective, it will be sufficient to show that for every prime ideal p of R, the localized map $F_p : A_p \# \bar{A}_p \to \operatorname{End}_{R_p}(A_p)$ is surjective. Therefore, we may restrict attention to the case where R is local; we will be done if we can show that, for all $h^* \in H^*$, $h \in H$, we have that $h^* \otimes h \in \operatorname{Im} F$.

If R is local, then the integral (cf. 1.6) is free of rank one, i.e. $\int = Rx^*$, for some $x^* \in H^*$. Then the map $v : H \to H^*$ introduced in Lemma 1.8 is a bijection. Consider $l = v^{-1}(h^*) \in H$, and take

$$\Gamma = \sum_{(l), v(l_{(2)})} h S(l_{(1)}) f^{-1}(v(l_{(2)})_{(1)}) \# S(f^{-1}(v(l_{(2)})_{(2)})) .$$

Then we have for all $k \in H$,

$$\begin{aligned} F(\Gamma)(k) &= \sum_{(l), v(l_{(2)}), (k)} \langle v(l_{(2)})_{(2)}, k_{(2)} \rangle h S(l_{(1)}) \\ &\qquad \times f^{-1}(v(l_{(2)})_{(1)}) S(f^{-1}(v(l_{(2)})_{(3)})) k_{(1)} \\ &= \sum_{(l), v(l_{(2)}), (k)} \langle v(l_{(2)})_{(2)}, k_{(2)} \rangle h S(l_{(1)}) \varepsilon(f^{-1}(v(l_{(2)})_{(1)})) k_{(1)} \\ &= \sum_{(l),(k)} \langle v(l_{(2)}), k_{(2)} \rangle k_{(1)} h S(l_{(1)}) \end{aligned}$$

$$= \sum_{(l)} \langle v(l_{(2)}), k \rangle l_{(1)} h S(l_{(2)}) \quad \text{(applying Lemma 1.8)}$$

$$= \langle v(l), k \rangle h = \langle h^*, k \rangle h = (h^* \otimes h)(k)\,.$$

From a rank argument, it follows that F is injective too. A similar argument shows that $G: \bar{A} \# A \to \mathrm{End}_R(A)^{\mathrm{opp}}$ is surjective: observe first that, for all $h,k,l \in H$,

$$G(h \# l)(k) = \sum_{(h),(k)} \langle f(S(k_{(2)})), h_{(2)} \rangle h_{(1)} k_{(1)} l\,.$$

Take $l = v^{-1}(h^*) \in H$, and let

$$\Gamma = \sum_{(l),v(l_{(2)})} S(f^{-1}(v(l_{(2)})_{(2)})) \# h S(l_{(1)}) f^{-1}(v(l_{(2)})_{(1)}) \in \bar{A} \# A\,;$$

we leave it to the reader to show that $G(\Gamma) = h^* \otimes h$—the proof uses the fact that $f^* = f$. $\square$

Note. If $H = RG$, then the proof of the foregoing lemma becomes much easier. It then suffices to construct an inverse image under F for every $v_\nu \otimes u_\tau \in (RG)^* \otimes RG$. This inverse image Γ is given by $\Gamma = \sum_{\alpha \in G} u_{\tau\nu^{-1}\alpha} \# r_\alpha u_{\alpha^{-1}}$, where $\sum_{\alpha \in G} r_\alpha u_\alpha = f^{-1}(v_\nu)$. We leave the verification to the reader.

5.3. Lemma. $\beta_A = (f^{-1} \otimes f) \circ t = \lambda$.

Proof. Let $\mathcal{H} = \mathrm{Hom}_R(J, H)$, $\mathcal{A} = \mathrm{Hom}_R(J, A)$, as previously. Define $u \in A$ by $u(h \otimes h^*) = hS(f^{-1}(h^*))$. First we will investigate the action and coaction of $\mathcal{H}$ on u; then we will show that $u \in I_{F(I)}$.

$$\chi_{\mathcal{A}}(u) = u \otimes u\,, \tag{14}$$

where u is viewed as an element of $\mathcal{A}$ in the first, and as an element of $\mathcal{H}$ in the second factor. (14) follows immediately, if we remark that $\chi_{\mathcal{A}} = \Delta_H$, and from the fact that f, and therefore u, are coalgebra homomorphisms, such that, by Proposition 1.5, u is grouplike in $\mathcal{H}$.

Define $\psi \in \mathrm{Hom}_R(J, H^*) = \mathcal{H}^*$ by $\psi(h \otimes h^*) = h^* f(S(h))$. Then for all $\mu \in \mathcal{H}$,

$$\mu \rightharpoonup u = \langle \psi, \mu \rangle * u\,. \tag{15}$$

Take $h \otimes h^* \in J$. Then

$$(\mu \rightharpoonup u)(h \otimes h^*)$$
$$= \sum_{(h),(h^*)} \mu(h_{(1)} \otimes h^*_{(1)}) \rightharpoonup u(h_{(2)} \otimes h^*_{(2)})$$

$$= \sum_{(h),(h^*)} \mu(h_{(1)} \otimes h^*_{(1)}) \rightarrow (h_{(2)} S(f^{-1}(h^*_{(2)})))$$

$$= \sum_{(h),(h^*)} \langle f(S(\mu(h_{(1)} \otimes h^*_{(1)}))), h_{(2)} S(f^{-1}(h^*_{(2)})) \rangle h_{(3)} S(f^{-1}(h^*_{(3)}))$$

$$= \sum_{(h),(h^*)} \langle \mu(h_{(1)} \otimes h^*_{(1)}), f(S(h_{(2)})) h^*_{(2)} \rangle h_{(3)} S(f^{-1}(h^*_{(3)}))$$

$$= \langle \mu, \psi \rangle * u .$$

$$u \in I_{F(I)} . \tag{16}$$

From Lemma 2.3, it follows that it is sufficient to show that, for all $j = h \otimes h^* \in J$ and $h \in H$ we have

$$\sum_{(h),(k),(k^*)} \langle h_{(1)}, \psi(k_{(1)} \otimes k^*_{(1)}) \rangle u(k_{(2)} \otimes k^*_{(2)}) h_{(2)}$$

$$= \sum_{(k),(k^*)} \rho(k_{(1)} \otimes k^*_{(1)})(h) u(k_{(2)} \otimes k^*_{(2)}) .$$

The left-hand side is equal to

$$\sum_{(h),(k),(k^*)} \langle h_{(1)}, k^*_{(1)} S(f(k_{(1)})) \rangle k_{(2)} S(f^{-1}(k^*_{(2)})) h_{(2)};$$

the right-hand side equals

$$\sum_{(h),(k),(k^*)} \langle f(S(k_{(1)})), h_{(1)} \rangle \langle k^*_{(1)}, h_{(2)} \rangle h_{(3)} k_{(2)} S(f^{-1}(k^*_{(2)})) ,$$

and (16) follows.

From (14)–(16), it follows that

$$\alpha_A(h \otimes h^*) = \sum_{(h),(h^*)} h_{(1)} S(f^{-1}(h^*_{(2)})) \otimes h^*_{(2)} S(f(h_{(2)})) ,$$

and therefore

$$\beta_A(h \otimes h^*) = (I * S \circ \alpha_A)(h \otimes h^*) = f^{-1}(h^*) \otimes f(h)$$

or $\beta_A = \lambda$. $\square$

5.4. Note. *If $f : H \rightarrow H^*$ is a* Hopf algebra isomorphism which is not selfdual, then we still have that $\lambda = (f^*)^{-1} \otimes f \in O(R, H)_{\min}$. But is it still true that $\lambda \in \operatorname{Im} \beta$? This question depends on the surjectivity of the map δ discussed in [13]:

$$\delta : H^2(H, R) \rightarrow P_{\mathrm{sk}}(H, R) ,$$

defined by

$$\langle \delta(g)(h), k\rangle = \sum_{(h),(k)} g(h_{(1)} \otimes k_{(1)})g^{-1}(k_{(2)} \otimes h_{(2)}) ,$$

for all $h,k \in H$. Here $H^2(H, R)$ is Sweedler's cohomology group, g^{-1} is the convolution inverse of g, and

$$P_{\mathrm{sk}}(H, R) = \{\phi \in \mathrm{Hopf}(H, H^*): \\ \sum_{(h)} \langle h_{(1)}, \phi(h_{(2)})\rangle = \varepsilon(h) \text{ for all } h \in H\} .$$

The map $d = f^* * S \circ f : H \to H^*$ lies in $P_{\mathrm{sk}}(H, R)$. Suppose that $d \in \mathrm{Im}\, \delta$, say $d = \delta([g])$, for some $g \in Z^2(H, R)$. We claim that this implies $\lambda \in \mathrm{Im}\, \beta$.

Consider the Sweedler smash product $A = R \#_g H$. As an R-module, A is equal to H, and the multiplication rule on A is given by

$$h.k = \sum_{(h),(k)} g(h_{(1)} \otimes k_{(1)})h_{(2)}k_{(2)} .$$

Define an action and coaction of H on A by $\psi_A = \Delta$ and

$$h \to k = \sum_{(k)} \langle f(S(h)), k_{(2)}\rangle k_{(1)} .$$

A modification of the proof of Theorem 5.1 shows that A is H-Azumaya and that $\beta_A = \lambda$.

6. Application to Orzech's subgroup of the Brauer–Long group

In [30], Orzech has introduced the following subgroup of $\mathrm{BD}(R, H)$: let $\theta : H \to H^*$ be a Hopf algebra map, and call an H-dimodule M a θ-module if the action and coaction of H on M are related by the formula

$$h \to m = \sum_{(m)} \langle \theta(m_{(1)}), h\rangle m_{(0)} .$$

An H-module algebra which is a θ-module is called a θ-algebra. Then

$$B_\theta(R, H) = \{\alpha \in \mathrm{BD}(R, H): \alpha \text{ is represented by a } \theta\text{-algebra}\}$$

is a subgroup of $\mathrm{BD}(R, H)$ (cf. [30]). The most simple case where θ is not trivial is the following: take $H = RC_2$, and define θ by $\langle \theta(\sigma), \sigma\rangle = -1$, where $C_2 = \{1, \sigma\}$. Then

$$B_\theta(R, H) = \mathrm{BW}(R)\,,$$

the Brauer–Wall group (cf. [3, 31, 36]). Another special case of this Brauer group is the Brauer group of Childs, Garfinkel and Orzech [18], and Knus [24]. More information on $B_\theta(R, H)$ may be found in [10, Section 5]. In Proposition 5.4 of [10], it was shown that $B_\theta^s(R, H) = B_\theta(R, H) \cap \mathrm{BD}^s(R, H)$ is a semi-direct product of $\mathrm{Br}(R)$ and $\mathrm{Gal}^s(R, H)$. Using the results of Sections 2 and 3, we obtain the following theorem:

6.1. Theorem. *We have an exact sequence*

$$1 \to B_\theta^s(R, H) \to B_\theta(R, H) \xrightarrow{\gamma} O_\theta(R, H)\,,$$

where $O_\theta(R, H) = \{ f \in \mathrm{Hopf}(H^*, H) \colon f * (S \circ f^* \circ \theta^* \circ f) \in P_{\mathrm{sk}}(H^*, R) \}$.

Proof. Obviously, the exact sequence of Theorem 3.8 restricts to

$$1 \to B_\theta^s(R, H) \to B_\theta(R, H) \xrightarrow{\beta} O(R, H)_{\min}\,.$$

Let A be an H-Azumaya algebra which is a θ-algebra, and consider $\alpha_A : J \to J$. Write

$$\mu = (I_H \otimes \varepsilon_{H^*}) \circ \alpha_A : J \to H\,, \qquad \mu^* = (\varepsilon_H \otimes I_{H^*}) \circ \alpha_A : J \to H^*\,.$$

Then for all $j \in J$,

$$\alpha_A(j) = \sum_{(j)} \mu(j_{(1)}) \otimes \mu^*(j_{(2)})\,.$$

Define $a_{11} : H \to H$, $a_{12} : H^* \to H$, $a_{21} : H \to H^*$, $a_{22} : H^* \to H^*$ by

$$a_{11}(h) = \mu(h \otimes 1)\,, \qquad a_{12}(h^*) = \mu(1 \otimes h^*)\,,$$

$$a_{21}(h) = \mu^*(h \otimes 1)\,, \qquad a_{22}(h^*) = \mu^*(1 \otimes h^*)\,,$$

for all $h \in H$, $h^* \in H^*$. Then

$$\alpha_A(h \otimes h^*) = \sum_{(h),(h^*)} a_{11}(h_{(1)}) a_{12}(h^*_{(1)}) \otimes a_{21}(h_{(2)}) a_{22}(h^*_{(2)})\,.$$

For all $h \in H$ and $a \in A$, we have

$$h \rightharpoonup m = \sum_{(m)} \langle \theta(m_{(1)}), h \rangle m_{(0)} = \sum_{(m)} \langle \theta^*, m_{(1)} \rangle m_{(0)}\,,$$

and therefore $\rho(h\otimes 1)=\rho(1\otimes\theta^*(h))$, and $\alpha_A(h\otimes 1)=\alpha_A(1\otimes\theta^*(h))$, or

$$\sum_{(h)} a_{11}(h_{(1)})\otimes a_{21}(h_{(2)})=\sum_{(h)} a_{12}(\theta^*(h_{(1)}))\otimes a_{22}(\theta^*(h_{(2)}))\,.$$

Applying $1\otimes\varepsilon$ and $\varepsilon\otimes 1$ to both sides, it follows that

$$a_{11}=a_{12}\circ\theta^*\,,\qquad a_{21}=a_{22}\circ\theta^*\,.$$

Next, take $u\in I_{F(I)}$. Recall that $\chi_{\mathscr{A}}(u)=u\otimes\mu$, $\gamma\rightarrow u=\langle\mu^*,\gamma\rangle * u$, for all $\gamma\in\mathscr{H}=\mathrm{Hom}_R(J,H)$. In particular,

$$I\rightarrow u=\langle\mu^*,I\rangle * u\,.$$

Now $\theta: H\rightarrow H^*$ extends to $\Theta:\mathscr{H}\rightarrow\mathscr{H}^*$, as follows: for $\gamma\in\mathscr{H}$,

$$\Theta(\gamma)=\theta\circ\gamma\in\mathscr{H}^*=\mathrm{Hom}_R(J,H^*)\,.$$

Since $\mathscr{A}$ is a Θ-algebra (by extension of scalars), we have that

$$I\rightarrow u=\langle\Theta(\mu),I\rangle * u\,,$$

and it follows that $\theta\circ\mu=\mu^*$. Therefore, we have for all $h^*\in H^*$

$$\theta(\mu(1\otimes h^*))=\mu^*(1\otimes h^*)\,,\qquad \theta(a_{12}(h^*))=a_{22}(h^*)\,,$$

hence

$$a_{22}=\theta\circ a_{12}\,.$$

Write $f=a_{12}\in\mathrm{Hopf}(H^*,H)$. Then f determines α_A and β_A completely since

$$a_{11}=f\circ\theta^*\,,\quad a_{21}=\theta\circ f\circ\theta^*\,,\quad a_{22}=\theta\circ f\,.$$

Now

$$\alpha_A(1\otimes h^*)=\sum_{(h^*)} a_{12}(h^*_{(1)})\otimes a_{22}(h^*_{(2)})=\sum_{(h^*)} f(h^*_{(1)})\otimes\theta(f(h^*_{(2)}))\,,$$

hence

$$\beta_A(1\otimes h^*)=\sum_{(h^*)} S(f(h^*_{(1)}))\otimes h^*_{(2)}S(\theta(f(h^*_{(3)})))\,.$$

Since $\beta_A \in O(R, H)_{\min}$, we have that

$$\begin{aligned} q(1 \otimes h^*) &= \langle h^*, 1\rangle = \varepsilon_{H^*}(h^*) \\ &= \sum_{(h^*)} \langle h^*_{(2)} S(\theta(f(h^*_{(3)}))), S(f(h^*_{(1)}))\rangle \\ &= \sum_{(h^*)} \langle (I * S \circ \theta \circ f)(h^*_{(1)}), S(f(h^*_{(2)}))\rangle \\ &= \sum_{(h^*)} \langle h^*_{(1)}, ((S \circ f) * (f^* \circ \theta^* \circ f))(h^*_{(2)})\rangle \,. \end{aligned}$$

It follows that $(S \circ f) * (f^* \circ \theta^* \circ f)$, and therefore $f * (S \circ f^* \circ \theta^* \circ f)$ are skew. If we define $\gamma([A]) = f$, then $f \in O_\theta(R, H)$.

We leave it to the reader to show that, if $\gamma([A]) = f$, $\gamma([B]) = g$, then

$$\gamma([A \# B]) = f * g * (S \circ f \circ (\theta * \theta^*) \circ g)\,.$$

Finally, if $\gamma([A]) = 0$, then $\beta([A]) = I_J$, hence $[A] \in B^s_\theta(R, H)$. □

7. Example: The free Hopf algebra of rank two

Let R be a commutative domain of characteristic different from 2, and fix $a, b \in R$ such that $ab = 2$. Let H_a be the Hopf algebra defined by

$$H_a = R[x]/(x^2 - ax)\,, \quad \Delta(x) = x \otimes 1 + 1 \otimes x - bx \otimes x\,, \quad \varepsilon(x) = 0\,.$$

For more details about this Hopf algebra and its Galois objects, we refer to [7, 23, 34]. We have that $H_a^* = H_b = R[y]/(y^2 - by)$, with $\langle x, y\rangle = -1$.

$\mathrm{BD}^s(R, H_a)$ has been computed explicitly in Proposition 4.1 of [10]. In 4.5 of the same paper, we suggested the following result:

7.1. Proposition. *With notations as above, and under the assumption that* 2 *is not invertible in* R, *we have*

$$\mathrm{BD}(R, H_a) = \mathrm{BD}^s(R, H_a) \qquad \text{if } a \neq \sqrt{2}\,, \tag{17}$$

$$\mathrm{BD}(R, H_a)/\mathrm{BD}^s(R, H_a) = \mathbb{Z}/2\mathbb{Z} \quad \text{if } a = \sqrt{2}\,. \tag{18}$$

Proof. If $a \neq \sqrt{2}$, then $a \neq b$, so $J = H_a \otimes H_b$ has only one Hopf algebra automorphism, namely the identity. A fortiori $O(R, H_a)_{\min} = \{1\}$, and (17) follows from Theorem 3.8.

If $a = \sqrt{2}$, then $a = b$, so $J = H_a \otimes H_a$. Then the switch map is an Hopf algebra automorphism, and it is easily seen that it lies in $O(R, H_a)_{\min}$. Therefore, $O(R, H_a)_{\min} = \{1, t\} = \mathbb{Z}/2\mathbb{Z}$, and from Theorem 4.1, it follows that β is onto. (18) then follows. □

7.2. Corollary. $\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) = D_8$.

Proof. It is well known that $\mathrm{Gal}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) = \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Br}(\mathbb{Z}[\sqrt{2}]) = \mathbb{Z}/2\mathbb{Z}$, so from [10, 4.1], it follows that $\mathrm{BD}^s(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}})$ contains 8 elements. From (18), it follows that $\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}})$ contains 16 elements. Consider the map

$$\mathrm{BD}(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) \to \mathrm{BD}(\mathbb{R}, \mathbb{R}C_2) .$$

Both groups contain 16 elements, and it is easily seen that the 16 different ones of the first group map to the 16 different ones of the second. The result then follows from the fact that $\mathrm{BD}(\mathbb{R}, \mathbb{R}C_2) = D_8$ (cf. [20] or [27]). □

7.3. Corollary. *Let* $\theta : H_{\sqrt{2}} \to H_{\sqrt{2}} = (H_{\sqrt{2}})^*$ *be the identity. Then* $B_\theta(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) = \mathbb{Z}/8\mathbb{Z}$.

Proof. From [10, Proposition 5.4], it follows that $B_\theta^s(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}})$ contains 4 elements. It may be verified easily that the 'minus' algebra constructed in Theorem 5.1 (or in [10, 4.3]) is a θ-algebra. Hence $B_\theta(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}})$ contains 8 elements. Now

$$B_\theta(\mathbb{Z}[\sqrt{2}], H_{\sqrt{2}}) \to \mathrm{BW}(R)$$

is an isomorphism, for arguments similar to the ones used in Corollary 7.2. It is well known that $\mathrm{BW}(\mathbb{R}) = \mathbb{Z}/8\mathbb{Z}$ (cf. e.g. [20, 36]). □

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